# Higher structures and quantization

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## 1. Motivation and overview

- Derived algebraic geometry (DAG) is a higher categorical enhancement of ordinary algebraic geometry that is useful for studying
  - (i) intersections of non-transversal subschemes
  - (ii) quotients by non-free group actions

- The building blocks of algebraic geometry are affine schemes

$$\mathbf{Aff} := \mathbf{CAlg}^{op} \qquad (\text{work over } \mathbb{K} = \text{field of char } 0).$$

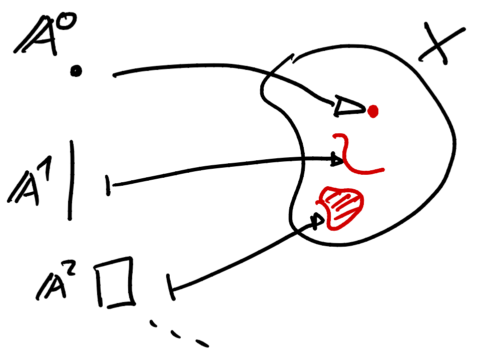
Given $A \in \mathbf{CAlg}$, write $\operatorname{Spec} A \in \mathbf{Aff}$ for associated affine scheme.

- For some purposes (e.g. mapping spaces), the category $\mathbf{Aff}$ is not big enough, so one has to introduce generalization

$$\underbrace{\mathbf{Aff}}_{\substack{\text{building}\\ \text{blocks}}} \xhookrightarrow{\text{Yoneda}} \underbrace{\mathbf{Sh}(\mathbf{Aff}, \mathbf{Set})}_{\substack{\text{generalized spaces}\\ \text{one gets from glueing}}}$$

Terminology: $X : \mathbf{Aff}^{op} = \mathbf{CAlg} \longrightarrow \mathbf{Set}$ is functor of points.

The idea is to describe a space $X$ by specifying how all test spaces $\operatorname{Spec} A \in \mathbf{Aff}$ map into it:




- $X(\mathbb{A}^0) =$ "set of points of $X$"
- $X(\mathbb{A}^1) =$ "set of curves in $X$"
- $X(\mathbb{A}^2) =$ "set of planes in $X$" ...

$$\mathbb{A}^n = \operatorname{Spec} \mathbb{K}[x_1, \dots, x_n]$$

■ The enhancements of DAG are best understood from this viewpoint:

$$
\begin{tikzcd}
\mathrm{Aff}^{op} = \mathrm{CAlg} \arrow[rr, "\text{scheme}"] \arrow[drr, "\text{stack}"] \arrow[ddrr, "\infty\text{-stack}"] \arrow[dd, hook] & & \mathrm{Set} \arrow[d, hook] \\
& & \mathrm{Grpd} \arrow[d, hook, dotted] \\
\mathrm{dAff}^{op} := \mathrm{dgCAlg}_{\geq 0} \arrow[rr, "\boxed{\text{derived } (\infty\text{-}) \text{ stack}}"'] & & \infty\text{-}\mathrm{Grpd}
\end{tikzcd}
$$

improves intersections

improves quotients

■ What's new in DAG?

(1) Shifted symplectic and Poisson structures [PTVV, CPTVV, Pridham, ...]

The tangent and cotangent spaces of a derived stack $X \in \mathrm{dSt}$ are cochain complexes:

$$T_x X = \left( \cdots \xrightarrow{d} (T_x X)^{-1} \xrightarrow{d} (T_x X)^{0} \xrightarrow{d} (T_x X)^{1} \xrightarrow{d} \cdots \right) \in \mathrm{Ch}$$

$\rightsquigarrow$ room for symplectic and Poisson structures w/ coh. degree $n \in \mathbb{Z}$ !

Interesting examples:

(i) $\mathrm{dCrit}(f : X \to \mathbb{A}^1)$ carries $(-1)$-shifted symplectic structure

$\rightsquigarrow$ BV formalism

(ii) $T^*[X/G]$ carries 0-shifted symplectic structure

$\rightsquigarrow$ ordinary phase spaces, but with derived and stacky features

(iii) $B\mathfrak{g} = [*/\mathfrak{g}]$ carries 1 and 2-shifted Poisson structures [Safronov]

$\rightsquigarrow$ quasi Lie bialgebras and invariant tensors $(\mathrm{Sym}^2 \mathfrak{g})^{\mathfrak{g}}$

(2) Shifted deformation quantization [CPTVV, Pridham, ...]

Quantizations of shifted Poisson structures give interesting algebraic structures:

$$\left( X \in \mathrm{dAff} \text{ w/ } n\text{-shifted Poisson} \right) \overset{\text{quantize}}{\rightsquigarrow} \left( \mathcal{O}_\hbar(X) \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Ch}) \right)$$

take modules

$$\left( \mathcal{O}_\hbar(X)^{\mathrm{dgMod}} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{dgCat}) \right)$$

$$\left( X \in \mathrm{dSt} \text{ w/ } n\text{-shifted Poisson} \right) \overset{\text{quantize}}{\rightsquigarrow} \left( \mathrm{QCoh}_\hbar(X) \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{dgCat}) \right)$$

Interesting examples:

(i) $\mathcal{O}_\hbar(\mathrm{dCrit}(f)) \in \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Ch}) \rightsquigarrow$ BV quantization

(ii) $\mathrm{QCoh}_\hbar(T^*[X/G]) \in \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{dgCat}) \rightsquigarrow$ D-modules

(iii) $\mathrm{QCoh}_\hbar(B\mathfrak{g}) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{dgCat}) \rightsquigarrow$ quantum groups $U_q\mathfrak{g}$

## ■ Plan for this lecture series:

- Shifted symplectic and Poisson structures for derived affines
- Shifted symplectic and Poisson structures for derived quotient stacks
- Some examples of quantizations

## 2. Derived affines

■ **Def:** The category of <u>derived affines</u> $dAff := dgCAlg_{\geq 0}^{op}$ is the opposite category of CDGAs concentrated in positive <u>homological</u> degree:

$$A_\bullet = \left( A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} \cdots \right) \in dgCAlg_{\geq 0}$$

The latter is a model category w.r.t. the following maps:

(i) $f^*: A_\bullet \to B_\bullet$ is <u>weak-equivalence</u> if $H_\bullet(f^*): H_\bullet(A) \xrightarrow{\cong} H_\bullet(B)$

(ii) $f^*: A_\bullet \to B_\bullet$ is <u>fibration</u> if $f_n^*: A_n \twoheadrightarrow B_n$ surjective $\forall n \geq 1$

(iii) $f^*: A_\bullet \to B_\bullet$ is <u>cofibration</u> if retract of semi-free extension

[ Semi-free extension means a map in $dgCAlg_{\geq 0}$ of the form $A_\bullet \to A[\{x_i\}]_\bullet$, where $\{x_i\}$ are new generators of degree $|x_i| \geq 0$ for the underlying graded algebra. ]

■ **Example:** (Derived affines from intersections)

Derived intersection in $dAff$ $\Leftrightarrow$ Homotopy pushout in $dgCAlg_{\geq 0}$

$$\begin{tikzcd}
\mathrm{Spec}\,A_\bullet \overset{\mathbb{R}}{\underset{\mathrm{Spec}\,C_\bullet}{\times}} \mathrm{Spec}\,B_\bullet \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Spec}\,B_\bullet \arrow[d, "g"] \\
\mathrm{Spec}\,A_\bullet \arrow[r, "f"'] & \mathrm{Spec}\,C_\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
C_\bullet \arrow[r, "g^*"] \arrow[d, "f^*"'] & B_\bullet \arrow[d, dashed] \\
A_\bullet \arrow[r, dashed] & A_\bullet \overset{\mathbb{L}}{\underset{C_\bullet}{\otimes}} B_\bullet
\end{tikzcd}$$

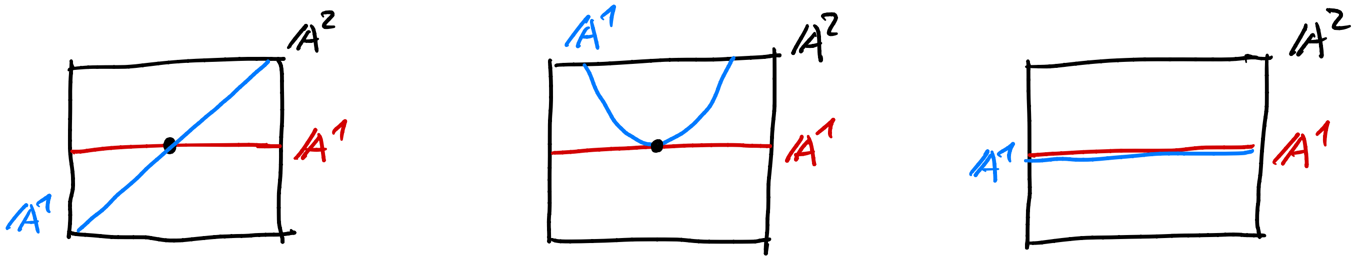
To compute the homotopy pushout we have to factorize

$$\begin{tikzcd}
C_\bullet \arrow[rr, "f^*"] \arrow[dr, "\text{cofib}", "\tilde{f}^*"'] & & A_\bullet \\
& \tilde{A}_\bullet \arrow[ur, "\text{w.e.}", "\sim"'] &
\end{tikzcd}$$

into a cofibration followed by a weak equivalence.

Then:
$$\boxed{A_\bullet \overset{\mathbb{L}}{\underset{C_\bullet}{\otimes}} B_\bullet \simeq \tilde{A}_\bullet \underset{C_\bullet}{\otimes} B_\bullet}$$

Let's study three simple examples of such objects:




① transversal ② multiplicity 2 ③ self-intersection

The red map is $f^*: C_\bullet = \mathbb{K}[x,y] \longrightarrow A_\bullet = \mathbb{K}[x]$

$$x \longmapsto x$$
$$y \longmapsto 0$$

Introducing $\tilde{A}_\bullet := \mathbb{K}[x, y, \xi]$ with $|y| = 0$, $|\xi| = 1$ and $\partial \xi = y$, we get

$$\begin{tikzcd}
\mathbb{K}[x,y] \arrow[rr, "f^*"] \arrow[dr, "\text{cofib}", "\tilde{f}^*"'] & & \mathbb{K}[x] \\
& \mathbb{K}[x,y,\xi] \arrow[ur, "\text{w.e.}", "\sim"'] &
\end{tikzcd}$$

with

$$\mathbb{K}[x,y,\xi] \longrightarrow \mathbb{K}[x]$$
$$x \longmapsto x$$
$$y, \xi \longmapsto 0$$

weak equivalence

$$\tilde{f}^*: \mathbb{K}[x,y] \longrightarrow \mathbb{K}[x,y,\xi]$$
$$x \longmapsto x$$
$$y \longmapsto y$$

semi-free extension $\Rightarrow$ cofibration

So we compute:

① $g^*: \mathbb{K}[x,y] \longrightarrow \mathbb{K}[z]$, $x \longmapsto z$, $y \longmapsto z$

$$A_\bullet \overset{\mathbb{L}}{\underset{C_\bullet}{\otimes}} B_\bullet \simeq \mathbb{K}[x,y,\xi] \underset{\mathbb{K}[x,y]}{\otimes} \mathbb{K}[z]$$
$$\cong \mathbb{K}[z,\xi] \quad \text{with } \partial \xi = z$$
$$\simeq \mathbb{K} \quad \text{(that's a point!)}$$

② $g^*: \mathbb{K}[x,y] \longrightarrow \mathbb{K}[z]$, $x \longmapsto z$, $y \longmapsto z^2$

$$A_\bullet \overset{\mathbb{L}}{\underset{C_\bullet}{\otimes}} B_\bullet \simeq \mathbb{K}[x,y,\xi] \underset{\mathbb{K}[x,y]}{\otimes} \mathbb{K}[z]$$
$$\cong \mathbb{K}[z,\xi] \quad \text{with } \partial \xi = z^2$$
$$\simeq \mathbb{K}[z]/(z^2) \quad \text{(that's singular in CAlg)}$$

③ $g^*: \mathbb{K}[x,y] \longrightarrow \mathbb{K}[z]$, $x \longmapsto z$, $y \longmapsto 0$

$$A_\bullet \overset{\mathbb{L}}{\underset{C_\bullet}{\otimes}} B_\bullet \simeq \mathbb{K}[x,y,\xi] \underset{\mathbb{K}[x,y]}{\otimes} \mathbb{K}[z]$$
$$\simeq \mathbb{K}[z,\xi] \quad \text{with } \partial \xi = 0$$

$\Rightarrow$ non trivial homology in deg 0 and 1

■ The usual construction of Kähler differentials (i.e. 1-forms) from algebraic geometry generalizes (with some subtleties discussed below) to derived affines.

Recall: Let $A_\bullet \in \mathrm{dgCAlg}_{\geq 0}$ and $M_\bullet \in {}_{A_\bullet}\mathrm{dgMod}$. A derivation of degree $n$ is a linear map $D: A_\bullet \to M_\bullet$ of degree $n$ that satisfies the graded Leibniz rule $D(a\,a') = (Da)\,a' + (-1)^{|a|\,n}\, a\,(Da')$ $\forall a,a' \in A_\bullet$.

The derivations assemble into an $A_\bullet$-dg-module $\mathrm{Der}(A_\bullet, M_\bullet)$ with:

(i) Differential: $\partial D := \partial_M \circ D - (-1)^{|D|}\, D \circ \partial_A$

(ii) Module structure: $(a \cdot D)(a') := a\, D(a')$ $\forall a,a' \in A_\bullet$

■ Def: Let $A_\bullet \in \mathrm{dgCAlg}_{\geq 0}$. The $A_\bullet$-dg-module $\Omega^1_{A_\bullet} \in {}_{A_\bullet}\mathrm{dgMod}$ of Kähler differentials on $A_\bullet$ is defined as the corepresenting object

$$\mathrm{Der}(A_\bullet, -) \cong \underline{\hom}_{A_\bullet}(\Omega^1_{A_\bullet}, -) : {}_{A_\bullet}\mathrm{dgMod} \longrightarrow \mathrm{Ch}$$

The cotangent complex of $A_\bullet$ is defined by

$$\mathbb{L}_{A_\bullet} := A_\bullet \underset{\tilde{A}_\bullet}{\otimes} \Omega^1_{\tilde{A}_\bullet} \in {}_{A_\bullet}\mathrm{dgMod},$$

where $\tilde{A}_\bullet \xrightarrow{\sim} A_\bullet$ is a cofibrant resolution of $A_\bullet$. (In practice, $\tilde{A}_\bullet$ is semi-free.)

■ Warning: When working with $\mathbb{L}_{A_\bullet}$ you have to make sure that all your subsequent definitions and constructions do not depend on the choice of resolution $\tilde{A}_\bullet \xrightarrow{\sim} A_\bullet$ $\rightsquigarrow$ use methods from model category theory.

■ Example: (Kähler differentials vs cotangent complex)

Let's compare for our three examples above:

① $\mathbb{K}$ is cofibrant, so we can take $\mathbb{K} \xrightarrow{\mathrm{id}} \mathbb{K}$ as resolution. Then

$$\mathbb{L}_{\mathbb{K}} = \Omega^1_{\mathbb{K}} = 0 \qquad \text{(as expected for point)}$$

② $\mathbb{K}[z]/(z^2)$ is <u>not</u> cofibrant. We use the resolution $\underset{\partial\xi = z^2}{\mathbb{K}[z,\xi]} \xrightarrow{\sim} \mathbb{K}[z]/(z^2)$.

We get for the Kähler differentials

$$\Omega^1_{\mathbb{K}[z]/(z^2)} = \frac{\mathbb{K}[z]}{(z^2)} dz \Big/ (z\,dz) \cong \mathbb{K}\,dz$$

with module structure given by $z.dz = 0$.

That's <u>not</u> projective, hence <u>not</u> dualizable, hence no vector fields.

On the other hand, the cotangent complex

$$\mathbb{L}_{\mathbb{K}[z]/(z^2)} = \frac{\mathbb{K}[z]}{(z^2)} \underset{\mathbb{K}[z,\xi]}{\otimes} \left( \mathbb{K}[z,\xi]\,dz \oplus \mathbb{K}[z,\xi]\,d\xi \right)$$

$$\cong \left( \overset{(0)}{\frac{\mathbb{K}[z]}{(z^2)}} dz \oplus \overset{(1)}{\frac{\mathbb{K}[z]}{(z^2)}} d\xi \right)$$

Slogan: $\mathbb{K}[z]/(z^2)$ is singular in AG, but smooth in DAG

is semi-free as a module, hence dualizable, hence there are vector fields.

Convention: Differentials anticommute $\partial d + d\partial = 0$

$$\Rightarrow \quad \partial(d\xi) = -d(\partial\xi) = -d(z^2) = -2z\,dz$$

Note that $H_\bullet\left(\mathbb{L}_{\mathbb{K}[z]/(z^2)}\right) \cong \left( \overset{(0)}{\mathbb{K}\,dz} \oplus \overset{(1)}{\mathbb{K}\,z\,d\xi} \right)$ is non-trivial in degree 0 and 1

③ Exercise.

■ The cotangent complex can be extended to a de Rham double complex.

Our sign conventions for chain cochain (double) complexes $(V_\bullet^\bullet, \partial, \delta) \in Ch_\bullet^\bullet$ are that of anticommuting differentials $\partial\delta + \delta\partial = 0$, which is compatible with the totalized Koszul signs (i.e. $|v| = i - j$ for $v \in V_j^i$)

$$\Rightarrow \quad \tau : V_\bullet^\bullet \otimes W_\bullet^\bullet \longrightarrow W_\bullet^\bullet \otimes V_\bullet^\bullet$$
$$v \otimes w \longmapsto (-1)^{|v||w|}\, w \otimes v$$

■ Def: Let $A_\bullet \in \mathrm{dgCAlg}_{\geq 0}$. The de Rham algebra of $A_\bullet$ is defined as the chain cochain CDGA

$$DR^\bullet(A_\bullet) := \mathrm{Sym}_{\tilde{A}_\bullet}\left(\Omega^{1}_{\tilde{A}_\bullet}[-1]\right) \in \mathrm{DGdgCAlg}^{\geq 0}_{\geq 0},$$

where $\tilde{A}_\bullet \xrightarrow{\sim} A_\bullet$ is a cofibrant resolution. The cochain differential $d$ is defined by extending $d: \tilde{A}_\bullet \to \Omega^1_{\tilde{A}_\bullet}$ to a square-zero derivation.

■ Example: (Semi-free CDGAs)

If $A_\bullet = \mathbb{K}[x_1, \dots, x_n]$ is semi-free with generators of degree $|x_i| \geq 0$, then it is also cofibrant. The associated de Rham algebra

$$DR^\bullet(A_\bullet) = \mathbb{K}[x_1, \dots, x_n, dx_1, \dots, dx_n] \in \mathrm{DGdgCAlg}^{\geq 0}_{\geq 0}$$

is semi-free over $x_i$ of bidegree $\binom{0}{|x_i|}$ and $dx_i$ of bidegree $\binom{1}{|x_i|}$.

■ The concept of closed p-forms in DAG is rich and interesting: Simply demanding $d\omega = 0$ for $\omega \in DR^p(A_\bullet)$ isn't stable under weak equivalences, hence closed p-forms would depend on the resolution $\tilde{A}_\bullet \xrightarrow{\sim} A_\bullet$. That's of course non-sense, so we need a better homotopical definition.

■ Def: For $p \in \mathbb{Z}^{\geq 0}$, define the totalized complex

$$F^p DR^\bullet(A_\bullet) := \left(\left\{\prod_{i \geq p} DR^i(A_\bullet)_{i-j}\right\}_{j \in \mathbb{Z}}, d_{tot} = d + \partial\right).$$

An n-shifted closed p-form on $A_\bullet$ is a $(p+n)$-cocycle, i.e.

$$\omega \in \left(F^p DR^\bullet(A_\bullet)\right)^{p+n} \quad \text{s.t.} \quad d_{tot}\,\omega = 0$$

- Let's spell this out explicitly:

$\omega$ is a family of forms $\left(\omega_{(i)} \in DR^i(A_\bullet)_{i-p-n}\right)_{i\geq p}$ that satisfy

$$\partial \omega_{(p)} = 0 \quad \leftarrow \text{ homologically closed}$$

$$d\omega_{(p)} + \partial \omega_{(p+1)} = 0 \quad \leftarrow \text{ dR-closed up to } \partial\text{-homotopy}$$

$$d\omega_{(p+1)} + \partial \omega_{(p+2)} = 0$$

$$\vdots$$

- Def: Suppose that $A_\bullet \in dgCAlg_{\geq 0}$ has a dualizable cotangent complex $\mathbb{L}_{A_\bullet} \in {}_{A_\bullet}dgMod$. Denote its dual by $\mathbb{T}_{A_\bullet} \in {}_{A_\bullet}dgMod$ (called tangent complex).

An n-shifted symplectic structure on $A_\bullet$ is an n-shifted closed 2-form $\omega = \left(\omega_{(2)}, \omega_{(3)}, \ldots\right) \in \left(F^2 DR^\bullet(A_\bullet)\right)^{2+n}$ that is non-degenerate in the sense that

$$\omega_{(2)} : \mathbb{T}_{A_\bullet} \xrightarrow{\sim} \mathbb{L}_{A_\bullet}[-n] \text{ is weak equivalence in } {}_{A_\bullet}dgMod.$$

- Example: (Ordinary affine schemes)

Let $A \in CAlg$ be smooth and finitely generated. Then $\mathbb{L}_A = \Omega^1_A$ and $\mathbb{T}_A = Der(A)$ are concentrated in degree 0. Non-degeneracy implies that there are no n-shifted symplectic structures for $n \neq 0$.
For $n=0$ one recovers the usual concept of symplectic structure on affine schemes.

**Example:** (Derived critical locus / BV formalism)

To simplify things, let $A = \mathbb{K}[x_1,\dots,x_n] \in \mathrm{CAlg} \subseteq \mathrm{dgCAlg}_{\geq 0}$ be an ordinary free CAlg. Consider any function

$$\left( f : \mathrm{Spec}\, A \longrightarrow \mathbb{A}^1 = \mathrm{Spec}\, \mathbb{K}[z] \right) \Leftrightarrow \left( f \in A \right) .$$

**Wanted:** The derived critical locus of $f$, i.e. the derived intersection

$$\begin{tikzcd} \mathrm{dCrit}(f) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Spec}\, A \arrow[d, "df"] \\ \mathrm{Spec}\, A \arrow[r, "0"'] & T^*\mathrm{Spec}\, A \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \mathrm{Sym}_A \mathbb{T}_A \arrow[r, "(df)^*"] \arrow[d, "0^*"'] & A \arrow[d, dashed] \\ A \arrow[r, dashed] & \mathcal{O}(\mathrm{dCrit}(f)) \end{tikzcd} \quad \left( \begin{array}{l} \text{homotopy} \\ \text{pushout} \\ \text{in } \mathrm{dgCAlg}_{\geq 0} \end{array} \right)$$

The involved maps read explicitly as:

$$0^* : \mathrm{Sym}_A \mathbb{T}_A = \mathbb{K}[x_1,\dots,x_n, \partial_{x_1},\dots,\partial_{x_n}] \longrightarrow A = \mathbb{K}[x_1,\dots,x_n]$$
$$x_i \longmapsto x_i$$
$$\partial_{x_i} \longmapsto 0$$

$$(df)^* : \mathrm{Sym}_A \mathbb{T}_A = \mathbb{K}[x_1,\dots,x_n, \partial_{x_1},\dots,\partial_{x_n}] \longrightarrow A = \mathbb{K}[x_1,\dots,x_n]$$
$$x_i \longmapsto x_i$$
$$\partial_{x_i} \longmapsto \partial_{x_i} f$$

As cofibrant replacement for $0^*$ we can take

$$\tilde{0}^* : \mathbb{K}[x_1,\dots,x_n, \partial_{x_1},\dots,\partial_{x_n}] \longrightarrow \mathbb{K}[x_1,\dots,x_n, \partial_{x_1},\dots,\partial_{x_n}, \xi^1,\dots,\xi^n] = \tilde{A}_\bullet$$
$$x_i \longmapsto x_i$$
$$\partial_{x_i} \longmapsto \partial_{x_i}$$

with $|\partial_{x_i}| = 0$, $|\xi^i| = 1$ and $\partial \xi^i = \partial_{x_i}$

This gives

$$\mathcal{O}(\mathrm{dCrit}(f)) \simeq \tilde{A}_\bullet \otimes_{\mathrm{Sym}_A \mathbb{T}_A} A \cong \mathbb{K}[x_1,\dots,x_n, \xi^1,\dots,\xi^n] \in \mathrm{dgCAlg}_{\geq 0}$$

with $\partial \xi^i = \partial_{x_i} f$

Derived critical loci come with canonical $(-1)$-shifted symplectic structures, which are induced by the canonical symplectic structure on $T^*\operatorname{Spec} A$ and the theory of derived Lagrangian intersections (PTVV).

In our example: $\omega = (\omega_{(p)})_{p\geq 2}$ with

$$\omega_{(2)} = \sum_{i=1}^{n} dx_i\, d\xi^i \quad , \quad \omega_{(p)} = 0 \quad \forall p>2 .$$

<u>Note:</u> $d\omega_{(2)} = 0$ by $d^2=0$ and

$$\partial\omega_{(2)} = -\sum_{i=1}^{n} dx_i\, \partial(d\xi^i) = \sum_{i=1}^{n} dx_i\, d(\overbrace{\partial\xi^i}^{=\partial_{x_i} f})$$

$$= \sum_{i,j=1}^{n} (\partial_{x_i}\partial_{x_j} f)\, dx_i\, dx_j = 0$$

- In the case where $A_\bullet \in dgCAlg_{\geq 0}$ has a dualizable cotangent complex $\mathbb{L}_A \in {}_{A_\bullet}dgMod$, one can also introduce a concept of $n$-shifted Poisson structures.

  For this one introduces the CDGA of shifted polyvectors

called weight degree

$$\widehat{Pol}(A_\bullet, n) := \prod_{i\geq 0} \mathrm{Sym}^{i}_{A_\bullet}\big(\mathbb{T}_{A_\bullet}[n+1]\big) \in dgCAlg$$

and extends the Lie bracket on $\mathbb{T}_{A_\bullet}$ to a $P_{n+2}$-algebra structure $[\cdot,\cdot]$ on $\widehat{Pol}(A_\bullet, n)$ (called Schouten-Nijenhuis bracket).

- <u>Def:</u> An <u>$n$-shifted Poisson structure</u> on $A_\bullet \in dgCAlg_{\geq 0}$ is an element of degree $-n-2$

$$\pi \in \Big(F^2\widehat{Pol}(A_\bullet, n)\Big)_{-n-2} = \prod_{i\geq 2} \mathrm{Sym}^{i}_{A_\bullet}\big(\mathbb{T}_{A_\bullet}[n+1]\big)$$

  that satisfies the Maurer-Cartan equation

$$\partial\pi + \tfrac{1}{2}[\pi,\pi] = 0$$

- Let's spell this out explicitly:

$\Pi$ is a family of polyvectors $\left(\Pi_{(i)} \in \mathrm{Sym}^i_{A_\bullet}\left(\mathbb{T}_{A_\bullet[n+1]}\right)_{-n-2}\right)_{i\geq 2}$ that satisfy

$$\partial \Pi_{(2)} = 0 \leftarrow \text{homologically closed}$$

$$\partial \Pi_{(3)} + \frac{1}{2}[\Pi_{(2)}, \Pi_{(2)}] = 0 \leftarrow \text{Jacobi up to } \partial\text{-homotopy}$$

$$\partial \Pi_{(4)} + [\Pi_{(3)}, \Pi_{(2)}] = 0 \leftarrow \text{higher Jacobi homotopies}$$

$$\vdots$$

- Theorem: (Melani, CPTVV, Pridham, ...)

The space of $n$-shifted symplectic structures is weakly homotopy equivalent to the space of non-degenerate $n$-shifted Poisson structures.

- Example: (Back to derived critical loci / BV formalism)

Recall $\mathcal{O}(d\mathrm{Crit}(f)) \simeq \mathbb{K}[x_1, \dots, x_n, \xi^1, \dots, \xi^n] \in \mathrm{dgCAlg}_{\geq 0}$

with $|x_i| = 0$, $|\xi^i| = 1$, $\partial \xi^i = \partial_{x_i} f$ and $(-1)$-shifted symplectic structure

$$\omega_{(2)} = \sum_{i=1}^n dx_i \, d\xi^i \quad , \quad \omega_{(p)} = 0 \quad \forall p > 2 .$$

A corresponding $(-1)$-shifted Poisson structure is given by

$$\Pi_{(2)} = \sum_{i=1}^n \partial_{x_i} \partial_{\xi^i} \quad , \quad \Pi_{(p)} = 0 \quad \forall p > 2 .$$

The corresponding shifted bracket $\{x_i, \xi^j\} = \delta_i^j$ is called the antibracket in the BV formalism.

# 3. Derived quotient stacks

- All of this machinery for derived affines can be extended to derived Artin stacks, and in particular to derived quotient stacks.

- Def: (Special case, but sufficient for us)

  Let $Y := \operatorname{Spec} B_\bullet \in \mathrm{dAff}$ be a derived affine scheme with an action $v: Y \times G \longrightarrow Y$ of a smooth affine group scheme $G = \operatorname{Spec} H \in \mathrm{Grp}(\mathrm{Aff})$. The corresponding derived quotient stack is defined as the $\infty$-colimit

$$[Y/G] := \infty\text{-colim}\left( Y \leftleftarrows Y \times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; Y \times G^2 \cdots \right) \in \mathrm{dSt}$$

with the face and degeneracy maps given by

$$d_i : Y \times G^n \longrightarrow Y \times G^{n-1}$$

$$(y, g_1, \dots, g_n) \longmapsto \begin{cases} (y \cdot g_1, g_2, \dots, g_n) & \text{for } i = 0 \\ (y, g_1, \dots, g_i \cdot g_{i+1}, \dots, g_n) & \text{for } i = 1, \dots, n-1 \\ (y, g_1, \dots, g_{n-1}) & \text{for } i = n \end{cases}$$

$$s_i : Y \times G^n \longrightarrow Y \times G^{n+1}$$

$$(y, g_1, \dots, g_n) \longmapsto (y, g_1, \dots, g_i, e, g_{i+1}, \dots, g_n) \qquad \text{for } i = 0, \dots, n$$

- Geometric structures that can be pulled back along general dAff-morphisms $f: \operatorname{Spec} A_\bullet \to \operatorname{Spec} B_\bullet$, e.g. functions, forms, bundles, etc., can be extended easily to derived stacks. The idea is to use the functor of points:

E.g.: $\left( h \in \mathcal{O}(X) \right) \iff \left( \text{compatible family of } h_f \in A_0,\ \forall f : \operatorname{Spec} A_\bullet \to X \right)$

- **Def:** • The *function dg-algebra* of a derived stack $X \in dSt$ is defined as

$$\mathcal{O}(X) := \underset{\operatorname{Spec} A_\bullet \to X}{\operatorname{holim}} (A_\bullet) \in DGCAlg^{\mathbb{Z}}$$ ← unbounded

• The *de Rham algebra* of $X$ is defined as

$$DR(X) := \underset{\operatorname{Spec} A_\bullet \to X}{\operatorname{holim}} \left( DR^\bullet(A_\bullet) \right) \in DGDGCAlg^{\geq 0, \mathbb{Z}}$$

• The SM-dg-category of *quasi-coherent modules* on $X$ is defined as

$$QCoh(X) := \underset{\operatorname{Spec} A_\bullet \to X}{\operatorname{holim}} \left( {}_{A_\bullet}dgMod \right) \in SMDGCat$$

- **Example:** For a derived quotient stack $[Y/G] = [\operatorname{Spec} B_\bullet / \operatorname{Spec} H]$, we have

$$\begin{aligned}
\mathcal{O}([Y/G]) &= \mathcal{O}\left( \infty\text{-colim}\left( Y \leftleftarrows Y \times G \leftleftarrows Y \times G^2 \cdots \right) \right) \\
&\simeq \operatorname{holim}\left( \mathcal{O}(Y) \rightrightarrows \mathcal{O}(Y \times G) \rightrightarrows \mathcal{O}(Y \times G^2) \cdots \right) \\
&= \operatorname{holim}\left( B_\bullet \rightrightarrows B_\bullet \otimes H \rightrightarrows B_\bullet \otimes H^{\otimes 2} \cdots \right) = \underbrace{N^\bullet(G, B_\bullet)}_{\text{normalized group cochains}}
\end{aligned}$$

Forms work similarly.

For QCoh, one finds with some efforts

$$QCoh([Y/G]) \simeq {}_{B_\bullet}dgMod^H \qquad (\text{equivariant dg-modules})$$

- **Warning!** $[Y/G]$ can almost never be reconstructed from $\mathcal{O}([Y/G])$, but in many cases it can be reconstructed from $QCoh([Y/G])$. (Lurie)

**Example:** $BG = [*/G]$ for $G$ reductive:

$$\mathcal{O}(BG) \simeq N^\bullet(G, \mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*) \quad \text{vs.} \quad QCoh(BG) \simeq dgRep(G) \not\simeq Ch = QCoh(*)$$

This will be important when we discuss quantization.

- For geometric structures that do <u>NOT</u> pull back well under general $f: \operatorname{Spec} A_\bullet \to \operatorname{Spec} B_\bullet$ things get more complicated. This case includes polyvectors, Poisson structures, quantizations, etc. CPTVV developed formal localization techniques to deal with these issues, but we are going to use Pridham's more practical approach.

- <u>Idea:</u> (Pridham, simplified to quotient stacks)

Given any derived quotient stack $[Y/G]$, the resolution by derived affines

$$Y \leftleftarrows Y \times G \,\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\, Y \times G^2 \cdots$$

has in degree $n \geq 0$ an action of $G^{n+1}$

$$v_n : (Y \times G^n) \times G^{n+1} \longrightarrow Y \times G^n$$

$$(y, g_1, \dots, g_n, g_0', \dots, g_n') \longmapsto \left( y \cdot g_0', \, g_0'^{-1} g_1 g_1', \dots, g_{n-1}'^{-1} g_n g_n' \right)$$

The face and degeneracy maps are equivariant relative to the group hom.

$$\tilde{d}_i : G^{n+1} \longrightarrow G^n, \quad (g_0, \dots, g_n) \longmapsto (g_0, \dots, \check{g}_i, \dots, g_n)$$

$$\tilde{s}_i : G^{n+1} \longrightarrow G^{n+2}, \quad (g_0, \dots, g_n) \longmapsto (g_0, \dots, g_i, g_i, \dots, g_n)$$

Taking degree-wise formal Lie algebra quotients

$$[Y/\mathfrak{g}] \leftleftarrows [Y \times G / \mathfrak{g} \oplus \mathfrak{g}] \,\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\, [X \times G^2 / \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}] \cdots \qquad (*)$$

defines an étale resolution of $[Y/G]$ by "stacky derived affines". Since the face and degeneracy maps are étale, functions, forms, polyvectors, Poisson structures, quantizations, etc., of $[Y/G]$ can be defined as compatible families of degree-wise structures on $(*)$.

Punchline: A geometric structure on a derived quotient stack decomposes into a family of algebraic structures on "stacky derived affines".

■ <u>Def:</u> The category of <u>stacky derived affines</u> $\mathrm{StdAff} := \left(\mathrm{DGdgCAlg}^{\geq 0}_{\geq 0}\right)^{op}$ is the opposite category of chain cochain CDGAs concentrated in positive homological and cohomological degrees:

$$A^\bullet_\bullet = \begin{pmatrix} \vdots & & \vdots & & \vdots & \\ \uparrow & & \uparrow & & \uparrow & \\ A^2_0 & \xleftarrow{\partial} & A^2_1 & \xleftarrow{\partial} & A^2_2 & \leftarrow \cdots \\ \delta\uparrow & & \delta\uparrow & & \delta\uparrow & \\ A^1_0 & \xleftarrow[\partial]{} & A^1_1 & \xleftarrow[\partial]{} & A^1_2 & \leftarrow \cdots \\ \delta\uparrow & & \delta\uparrow & & \delta\uparrow & \\ A^0_0 & \xleftarrow[\partial]{} & A^0_1 & \xleftarrow[\partial]{} & A^0_2 & \leftarrow \cdots \end{pmatrix}$$

Model structure: chain-level-wise

Shifted symplectic and Poisson structures are defined similarly to the case of dAff, but note that there is now an additional grading:

$$\overset{\text{form}}{\mathrm{DR}^\bullet}(A^{\overset{\text{stacky}}{\bullet}}_{\underset{\text{derived}}{\bullet}}) \qquad\qquad \overset{\text{weight}}{\widehat{\mathrm{Pol}}}(A^{\overset{\text{stacky}}{\bullet}}_{\underset{\text{derived}}{\bullet}})$$

■ <u>Example:</u> (CE-algebras)

Given $B_\bullet \in \mathrm{dgCAlg}_{\geq 0}$ with Lie algebra action $\varrho : \mathfrak{g} \longrightarrow \mathrm{Der}(B_\bullet)$, we can form the Chevalley-Eilenberg CDGA

$$\mathrm{CE}^\bullet(\mathfrak{g}, B_\bullet) = \left(\mathrm{Sym}\, \mathfrak{g}^{\vee[-1]}\right) \otimes B_\bullet \in \mathrm{DGdgCAlg}^{\geq 0}_{\geq 0} .$$

This is generated by $b \in B_\bullet$ of bidegree $\binom{0}{|b|}$ and a basis $\theta^a \in \mathfrak{g}^\vee$, which has bidegree $\binom{1}{0}$. The differentials read as

$$\partial b = \partial_B b \quad , \quad \delta b = \theta^a \, \varrho(t_a)(b)$$

$$\partial \theta^a = 0 \quad , \quad \delta \theta^a = -\tfrac{1}{2} f^a_{bc}\, \theta^b \theta^c$$

The formal Lie algebra quotient is defined by

$$\left[\mathrm{Spec}\, B_\bullet / \mathfrak{g}\right] := \mathrm{Spec}\, \mathrm{CE}^\bullet(\mathfrak{g}, B_\bullet) \in \mathrm{StdAff}$$

■ **Example:** (Derived critical loci / BV formalism for quotient stacks)

Let $f : [X/G] \longrightarrow \mathbb{A}^1$ be a function on an ordinary quotient stack $[X/G] = [\mathrm{Spec}\, A / \mathrm{Spec}\, H]$. One can prove (see Benini, Safronov, AS) that

$$d\mathrm{Crit}(f) \simeq [Z/G] \quad \text{is derived quotient stack, where}$$

$$\mathcal{O}(Z)_\bullet = \mathrm{Sym}_A\Big( \mathsf{T}_{A\,[-1]} \oplus \big(A \otimes \mathfrak{g}_{[-2]}\big)\Big) \in \mathrm{dgCAlg}_{\geq 0}$$

with differential $\partial a = 0$, $\partial v = i_v(df)$, $\partial t = - i_{\rho(t)}(\lambda)$ — tautological 1-form on $T^*X$

Taking the étale resolution

$$[Z/\mathfrak{g}] \leftleftarrows [Z \times G / \mathfrak{g} \oplus \mathfrak{g}] \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

one can work out an explicit decription of the (-1)-shifted symplectic structure on $d\mathrm{Crit}(f)$, which in simplicial degree 0 starts with

$$\overset{o}{\omega}_{(2)} = d\Big( \underbrace{\lambda_{T^*[-1]X}}_{= p^i dx_i \text{ in coordinates}} - \overbrace{\lambda_{T^*[-1]B\mathfrak{g}}}^{= t_a d\theta^a} \Big) = dp^i dx_i - dt_a\, d\theta^a \,, \quad \overset{o}{\omega}_{(p)} = 0 \;\; \forall p > 2$$

⇝ Generalization of BV formalism to group actions.

■ **Example:** (Shifted Poisson structures on $BG$)

For $BG = [*/G]$, we have étale resolution

$$[*/\mathfrak{g}] \leftleftarrows [G / \mathfrak{g} \oplus \mathfrak{g}] \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

that starts with the usual CE-algebra $\mathcal{O}([*/\mathfrak{g}]) = CE^\bullet(\mathfrak{g}, \mathbb{K})$.

From degree counting, one sees that there can only be non-trivial 1-shifted and 2-shifted Poisson structures on $[*/\mathfrak{g}]$.

Choosing a basis $\{\theta^a \in \mathfrak{g}^\vee\}$, they read as follows:

<u>1-shifted:</u> $\Pi = \underbrace{\Pi_c^{ab}\,\theta^c\,\partial_{\theta^a}\partial_{\theta^b}}_{\text{bivector } \Pi_{(2)}} + \underbrace{\Pi^{abc}\,\partial_{\theta^a}\partial_{\theta^b}\partial_{\theta^c}}_{\text{trivector } \Pi_{(3)}}$

The MC equation $\partial\Pi + \frac{1}{2}[\Pi,\Pi] = 0$ is equivalent to

$$\left(\begin{array}{l} \{\cdot,\cdot\}: \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathfrak{g}^\vee \\ \qquad \theta^a \otimes \theta^b \mapsto \Pi^{ab}_c\,\theta^c \\ \tilde{\Pi} := \Pi^{abc}\, t_a t_b t_c \in \mathrm{Sym}^3\mathfrak{g} \end{array}\right)$$

defining a quasi Lie bialgebra structure on $(\mathfrak{g},[\cdot,\cdot])$.

<u>2-shifted:</u> $\Pi = \underbrace{\Pi^{ab}\,\partial_{\theta^a}\partial_{\theta^b}}_{\text{bivector } \Pi_{(2)}}$

The MC equation $\partial\Pi + \frac{1}{2}[\Pi,\Pi]=0$ is equivalent to $\tilde{\Pi} = \Pi^{ab}\, t_a t_b \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$.

<u>Note:</u> These are exactly the semi-classical data from quantum group theory!

<u>Rem:</u> For globalization to $BG$, see Safronov.

<u>Example:</u> (Cotangent bundles over quotient stacks)

Let $[X/G] = [\mathrm{Spec}\,A/\,\mathrm{Spec}\,H]$ be ordinary quotient stack.

The cotangent bundle $T^*[X/G] \simeq [T^*X /\!/ G] \simeq [\mu^{-1}(0)/G]$

can be computed via derived symplectic reduction (see e.g. Safronov).

We have étale resolution

$$[\mu^{-1}(0)/\mathfrak{g}] \leftleftarrows [\mu^{-1}(0)\times G/\mathfrak{g}\oplus\mathfrak{g}] \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} \cdots$$

that starts with $\mathcal{O}([\mu^{-1}(0)/\mathfrak{g}]) = CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)$, where

$$\mathcal{O}(\mu^{-1}(0))_\bullet = \mathrm{Sym}_A\left(T_A \xleftarrow{\mu^*} A\otimes\mathfrak{g}_{[-1]}\right) \quad \text{with } \mu^*(t) = -\,i_{\underline{g(t)}}(\lambda)$$

tautological 1-form on $T^*X$

The canonical 0-shifted Poisson structure on $T^*[X/G]$ defines on $\mathrm{Tot}\, CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)$ the Poisson bracket $\{\cdot,\cdot\}^0$ given by

$$\{a,a'\}^0 = 0\ ,\quad \{v,a\}^0 = v(a)\ ,\quad \{v,v'\}^0 = [v,v']\ ,\quad \{t_a, \theta^b\}^0 = -\delta_a^b$$

$\forall\, a,a' \in A$, $v,v' \in T_A$. This can be globalized to $T^*[X/G]$, see Benini, Pridham, AS.

# 4. Quantization

- The quantization problem in DAG:

Affine quantization:

$$\begin{pmatrix} X = \mathrm{Spec}\, A_\bullet \in dSt \text{ with} \\ n\text{-shifted Poisson structure } \pi \end{pmatrix} \overset{\text{quantize}}{\rightsquigarrow} \begin{pmatrix} \mathbb{E}_{n+1}\text{-algebra } A_{\bullet\hbar} \text{ recovering} \\ A_\bullet \text{ and } \pi \text{ for } \hbar \to 0 \end{pmatrix}$$

Global quantization:

$$\begin{pmatrix} X \in dSt \text{ with} \\ n\text{-shifted Poisson structure } \pi \end{pmatrix} \overset{\text{quantize}}{\rightsquigarrow} \begin{pmatrix} \mathbb{E}_{n}\text{-monoidal dg-category } \mathrm{QCoh}(X)_\hbar \\ \text{recovering } \mathrm{QCoh}(X) \text{ and } \pi \text{ for } \hbar \to 0 \end{pmatrix}$$

- Rem: For $X = \mathrm{Spec}\, A_\bullet$ derived affine, each affine quantization gives a global one by taking dg-modules:

$$A_{\bullet\hbar} \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Ch}) \longmapsto A_{\bullet\hbar}\,\mathrm{dgMod} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{dgCat})$$

- Theorem:

The global quantization problem can be solved for $X$ derived Artin stack and

(i) $n \geq 1$ (CPTVV)

(ii) $n = 0$ (Pridham)

- Warning! These are abstract existence results, so don't expect any explicit formulas for the quantized algebras/categories.

The only explicit results I am aware of are:

(1) $X = \mathrm{Spec}(\mathrm{Sym}\, V_\bullet)$ with constant coefficient $(n \geq 0)$-shifted Poisson structure

$\rightsquigarrow$ Weyl n-algebras (Markarian)

(2) $X = \mathrm{Spec}(\mathrm{Sym}\, V_\bullet)$ with constant coefficient $(-1)$-shifted Poisson structure

$\rightsquigarrow$ BV quantization (Costello/Gwilliam and Gwilliam/Haugseng)

(3) $X = T^*[\mathrm{Spec}\, A / \mathrm{Spec}\, H]$ with canonical $0$-shifted Poisson structure

$\rightsquigarrow$ D-modules (Benini/Pridham/AS)

- Example: (BV quantization, simplest case)

Recall: Given $f : \mathrm{Spec}\, \mathbb{K}[x_1, \dots, x_n] \longrightarrow \mathbb{A}^1$, then

$$d\mathrm{Crit}(f) \simeq \mathrm{Spec}\, \mathbb{K}[x_1, \dots, x_n, \xi^1, \dots, \xi^n] \quad \text{with } |x_i| = 0,\ |\xi^i| = 1,\ \partial \xi^i = \partial_{x_i} f .$$

The canonical $(-1)$-shifted Poisson structure is $\pi_{(2)} = \sum_{i=1}^n \partial_{x_i} \partial_{\xi^i}$, $\pi_{(p)} = 0 \ \forall p > 2$.

$\Rightarrow$ $(-1)$-shifted Poisson bracket:

$$\{x_i, \xi^j\} = \delta_i^j , \qquad \{x_i, x_j\} = \{\xi^i, \xi^j\} = 0$$

An $\mathbb{E}_0$-quantization of this $P_0$-algebra is given by

$$A_{\bullet\hbar} = \Big( \mathbb{K}[x_1, \dots, x_n, \xi^1, \dots, \xi^n][[\hbar]],\ \partial_\hbar = \partial + \hbar\, \Delta_{BV} \Big) \quad \text{with } \Delta_{BV} = \sum_{i=1}^n \partial_{x_i} \partial_{\xi^i}$$

BV Laplacian

The shifted Poisson bracket is reconstructed from violation of $\partial_\hbar$ being a derivation:

$$\partial_\hbar (a \cdot b) = (\partial_\hbar a)\, b + (-1)^{|a|} a\, (\partial_\hbar b) + \hbar \{a, b\} \qquad \forall a, b \in A_{\bullet\hbar}$$

- Strategy for constructing a global quantization of derived quotient stacks $X = [Y/G] = [\operatorname{Spec} B_\bullet / \operatorname{Spec} H]$ along $(n\geq 0)$-shifted Poisson structures:

<u>Step 1:</u> Express data using étale resolution

$$\left([Y/\mathfrak{g}], \pi^0\right) \leftleftarrows \left(\left[Y\times G/_{\mathfrak{g}\oplus\mathfrak{g}}\right], \pi^1\right) \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \cdots$$

This gives a cosimplicial diagram of $\mathbb{P}_{n+1}$-algebras

$$\operatorname{Tot} CE^\bullet(\mathfrak{g}, B_\bullet) \rightrightarrows \operatorname{Tot} CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, B_\bullet\otimes H) \,\substack{\rightarrow\\\rightarrow\\\rightarrow}\, \cdots$$

<u>Step 2:</u> Construct compatible degree-wise quantizations to $\mathbb{E}_{n+1}$-algebras

$$\operatorname{Tot} CE^\bullet(\mathfrak{g}, B_\bullet)_\hbar \rightrightarrows \operatorname{Tot} CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, B_\bullet\otimes H)_\hbar \,\substack{\rightarrow\\\rightarrow\\\rightarrow}\, \cdots$$

and pass over to $\mathbb{E}_n$-monoidal dg-categories of modules

$$\left({}_{\operatorname{Tot} CE^\bullet(\mathfrak{g}, B_\bullet)_\hbar}\mathrm{dgMod} \rightrightarrows {}_{\operatorname{Tot} CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, B_\bullet\otimes H)_\hbar}\mathrm{dgMod} \,\substack{\rightarrow\\\rightarrow\\\rightarrow}\, \cdots\right) = (**)$$

<u>Step 3:</u> Determine global quantization by computing homotopy limit

$$\mathrm{QCoh}([Y/G])_\hbar := \operatorname{holim} (**) \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{dgCat})$$

- <u>Theorem:</u> (Benini / Pridham / AS)
For $T^*[X/G] \in \mathrm{dSt}$ with $X = \operatorname{Spec} A$ smooth and $G = \operatorname{Spec} H$ reductive, global quantization of canonical 0-shifted Poisson structure gives D-modules

$$\mathrm{QCoh}(T^*[X/G])_\hbar \simeq D_\hbar\text{-}\mathrm{Mod}([X/G]) .$$

This can be spelled out very explicitly:

**Objects:** Triples $(\mathcal{E}^\bullet, \nabla, \psi)$ consisting of

(i) $G$-eqv. $\mathcal{O}(X)[[\hbar]]$-dg-module $\mathcal{E}^\bullet$

(ii) $G$-eqv. dg-connection $\nabla : \mathcal{E}^\bullet \longrightarrow \Omega^1(X)[[\hbar]] \otimes_{\mathcal{O}(X)[[\hbar]]} \mathcal{E}^\bullet$

with respect to $\hbar\, d^{dR}$, i.e. $\nabla(a\,s) = \hbar\,(d^{dR}a) \otimes s + a\,\nabla(s)$

(iii) $G$-eqv. graded module map $\psi : \mathfrak{g}[1] \otimes \mathcal{E}^\# \longrightarrow \mathcal{E}^\#$

satisfying the following conditions

$$\nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar\, \nabla_{[v,v']} \qquad \text{(flat connection)}$$

$$\nabla_v \psi_t - \psi_t \nabla_v = 0$$

$$\psi_t \psi_{t'} + \psi_{t'} \psi_t = 0$$

$$\partial \psi_t + \psi_t \partial = \nabla_{\mu^*(t)} + \hbar\, \varrho(t)$$

**Morphisms:** $\underline{\hom}_{\mathcal{O}(X)[[\hbar]]}(\mathcal{E}^\bullet, \mathcal{E}'^\bullet)$ preserving $G, \nabla, \psi$

- **Work in progress:** (Kemp / Laugwitz / AS)

$$\left(\begin{array}{l} BG = [\ast / \mathrm{Spec} H] \text{ with} \\ \text{2-shifted Poisson structure} \end{array}\right) \xrightarrow{\text{expected}} \left(\begin{array}{l} \mathrm{QCoh}(BG)_\hbar \simeq \mathrm{dgRep}(G)_\hbar \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{dgCat}) \\ \text{quantum group dg-representations} \end{array}\right)$$

but how does this work in detail?

When $G$ is a 2-group (n-group), does one get interesting quantum 2-groups? (quantum n-groups)