From Fredenhagen’s universal algebra to homotopy theory and operads

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Quantum Physics meets Mathematics:
A workshop on the occasion of Klaus Fredenhagen’s 70th birthday,
University of Hamburg, December 8-9, 2017.
1. Fredenhagen’s universal algebra and some of its applications in AQFT

2. How it provided motivations for our homotopical AQFT program

3. A whole zoo of (improved) universal constructions from operad theory

4. Birthday present:
   A theorem about AQFTs on spacetimes with timelike boundary

Fredenhagen \(\rightarrow\) Homotopy \(\rightarrow\) quantum gauge theory?

Fredenhagen \(\rightarrow\) Operads \(\rightarrow\) boundary AQFT
Fredenhagen’s universal algebra
Starting late 80’s: Klaus studied representation theory and superselection sectors of 2-dim. QFTs, including chiral conformal QFTs [cf. Carpi’s talk].

A ccQFT has an underlying functor $\mathcal{A} : \text{Int}(S^1) \to \text{Alg}$, which assigns
- an algebra (of observables) $\mathcal{A}(I)$ to every proper interval $I \subset S^1$;
- a homomorphism $\mathcal{A}(i) : \mathcal{A}(I) \to \mathcal{A}(I')$ to every inclusion $i : I \to I'$.

**Def:** Fredenhagen’s universal algebra corresponding to $\mathcal{A} : \text{Int}(S^1) \to \text{Alg}$ is
- an algebra $\mathcal{A}^u \in \text{Alg}$;
- together with homs $\kappa_I : \mathcal{A}(I) \to \mathcal{A}^u$ satisfying $\kappa_{I'} \circ \mathcal{A}(i) = \kappa_I$, for all $i$, which are universal: For any other such $(B, \{\rho_I : \mathcal{A}(I) \to B\})$ there exists a unique hom $\rho^u : \mathcal{A}^u \to B$, such that the following diagrams commute

\[
\begin{array}{ccc}
\mathcal{A}(I) & \xrightarrow{\mathcal{A}(i)} & \mathcal{A}(I') \\
\downarrow{\kappa_I} & & \downarrow{\kappa_{I'}} \\
\mathcal{A}^u & \xrightarrow{\exists!} & B \\
\rho_I \quad \rho^u \quad \rho_{I'}
\end{array}
\]

$\mathcal{A}^u$ is useful for representation theory $\text{Rep}(\mathcal{A}) \cong \text{Rep}(\mathcal{A}^u)$. 
Intermezzo: Colimits

- The universal algebra is a special instance of a colimit in category theory.

**Def:**

(i) A cocone of a functor $F : D \to C$ is an object $c \in C$ together with a natural transformation $\xi : F \to \Delta(c)$ to the constant functor $\Delta(c) : d \mapsto c$.

(ii) A colimit of $F$ is a universal cocone $(\text{colim}F, \iota : F \to \Delta(\text{colim}F))$, i.e. given any other cocone $(c, \xi : F \to \Delta(c))$ there exists a unique $C$-morphism $f : \text{colim}F \to c$, such that the following diagram commutes:

\[
\begin{array}{ccc}
F & \xrightarrow{\xi} & \Delta(c) \\
\downarrow{\iota} & & \downarrow{\Delta(f)} \\
\Delta(\text{colim}F) & & \\
\end{array}
\]

- Fredenhagen’s definition is recovered by writing this in components:

\[
\begin{array}{ccc}
F(d) & \xrightarrow{\xi_d} & \text{colim}F - \xrightarrow{f} - c \\
\downarrow{\iota_d} & \iff & \downarrow{\iota_d'} \\
F(g) & \xrightarrow{\xi_{d'}} & \text{colim}F - \xrightarrow{f} - c \\
\downarrow{\iota_d} & & \downarrow{\iota_d'} \\
F(d') & & \\
\end{array}
\]

- **Good News:** For functors $\mathcal{A} : D \to \text{Alg}$ with values in algebras, the colimit $\text{colim}\mathcal{A}$ always exists! $\Rightarrow$ Fredenhagen’s universal algebra always exists!
Beyond the circle

- The universal algebra is a very flexible concept!
- **Problem:** Want to construct $\mathcal{A}(M)$ on complicated spacetime $M$, but just manage to get a functor $\mathcal{A} : \text{Reg}_M \to \text{Alg}$ on ‘nice’ regions $U \subseteq M$.
- **Solution:** Set $\mathcal{A}(M) := \text{colim}(\mathcal{A} : \text{Reg}_M \to \text{Alg})$ to be universal algebra!
- Together with students, Klaus studied particular applications:
  1. **Maxwell theory** [Benni Lang, Diplomarbeit 2010]
     Maxwell’s equations $dF = 0 = \delta F$ for $F \in \Omega^2(M)$ allow for topological charges $[F] \in H^2(M; \mathbb{R})$ and $[*F] \in H^{m-2}(M; \mathbb{R})$ on general spacetimes $M$.
     Construct $\mathcal{A} : \text{Reg}_M \to \text{Alg}$ for contractible regions $\text{Reg}_M$ in $M$ and analyze properties of the global algebra $\mathcal{A}(M) := \text{colim} \mathcal{A}$. [more on this later...]
  2. **Non-globally hyperbolic spacetimes** [Christian Sommer, Diplomarbeit 2006]
     Let $M$ be a spacetime with timelike boundary and consider globally hyperbolic regions $\text{Reg}_{\text{int}M}$ in the interior $\text{int}M \subseteq M$.
     Assuming $F$-locality [Kay], $\mathcal{A} : \text{Reg}_{\text{int}M} \to \text{Alg}$ can be constructed as usual.
     Relationship between ideals of $\mathcal{A}(M) := \text{colim} \mathcal{A}$ and boundary conditions!
Fredenhagen’s universal algebra in LCQFT

- In locally covariant QFT, one studies functors $\mathcal{A} : \text{Loc} \to \text{Alg}$ on the category of all spacetimes $\text{Loc}$ [cf. Fewster’s talk].

- Consider full subcategory $j : \text{Loc}_\circ \subseteq \to \text{Loc}$ of contractible spacetimes and assume that you are given a theory $\mathcal{A}_\circ : \text{Loc}_\circ \to \text{Alg}$.

- **Construction/Observation:** [Benni Lang, PhD in York (2014) with Fewster]
  
  - On every spacetime $M \in \text{Loc}$, we may compute the universal algebra

$$\mathcal{A}(M) := \text{colim} \left( \text{Loc}_\circ /M \xrightarrow{Q_M} \text{Loc}_\circ \xrightarrow{\mathcal{A}_\circ} \text{Alg} \right)$$

  on the over category $\text{Loc}_\circ /M$ of contractible regions $U \to M$ in $M$.

  - This defines functor $\mathcal{A} : \text{Loc} \to \text{Alg}$ on all spacetimes $\text{Loc}$, which is universal in the sense of left Kan extensions

\[
\begin{array}{ccc}
\text{Loc}_\circ & \xrightarrow{\mathcal{A}_\circ} & \text{Alg} \\
\downarrow^j & & \downarrow^\epsilon \\
\text{Loc} & \xrightarrow{\mathcal{A} \cong \text{Lan}_j \mathcal{A}_\circ} & \\
\end{array}
\]

**Universal algebra in LCQFT = left Kan extension along $j : \text{Loc}_\circ \to \text{Loc}$**
The homotopical AQFT program

M. Benini, AS, U. Schreiber, R. J. Szabo and L. Woike
Universal algebra for Maxwell theory

- Classical Maxwell theory on contractible spacetimes $U \in \text{Loc}_c$:
  - $A \in \Omega^1(U)$ with gauge trasfos $A \mapsto A + \frac{1}{2\pi i} d \log g$, for $g \in C^\infty(U, U(1))$
  - Maxwell’s equation $\delta dA = 0$, i.e. $A \in \Omega^1_{\delta d}(U)$

- Gauge invariant and on-shell exponential observables $[\varphi] \in \Omega^1_{c \delta}(U) / \delta d \Omega^1_c(U)$

  \[ \mathcal{O}_{[\varphi]} : \frac{\Omega^1_{\delta d}(U)}{d \Omega^0(U)} \to \mathbb{C}, \quad [A] \mapsto \exp \left( 2\pi i \int_U \varphi \wedge *A \right) \]

  with presymplectic structure $\omega_U([\varphi], [\varphi']) = \exp \left( 2\pi i \int_U \varphi \wedge *G^\square(\varphi') \right)$.

- Quantum Maxwell theory $\mathcal{A}_c : \text{Loc}_c \to \text{Alg}$ assigns the Weyl algebras.

- Universal algebra $\mathcal{A}(M)$ is the Weyl algebra corresponding to field strength theory on $M \in \text{Loc}$ [Dappiaggi,Lang]: $F \in \Omega^2(M)$ satisfying $\delta F = 0 = dF$

- **Problem:** $\mathcal{A}(M)$ does **NOT** have a gauge theoretic interpretation
  1. It misses flat connections/Aharonov-Bohm phases on $M$!
  2. $[F] \in H^2(M; \mathbb{R})$ is not integral $\Rightarrow$ No magnetic charge quantization!

Alexander Schenkel
Fredenhagen $\mapsto$ Homotopy & Operads
Klaus’ 70th Birthday (2017)
Homotopical improvement of the universal algebra

- **Important lesson:** Do **NOT** quotient out the gauge symmetries naively!

- Chain complex of $U(1)$-gauge fields on $U \in \text{Loc}_\odot$

$$\mathcal{F}_\odot(U) := \left( \Omega^1(U) \xrightarrow{\frac{1}{2\pi i} \, d \log} C^\infty(U, U(1)) \right)$$

- Smooth Pontryagin dual chain complex of observables on $U \in \text{Loc}_\odot$

$$\mathcal{O}_\odot(U) := \left( \Omega^{m-1}_c(U) \xrightarrow{d} \Omega^m_{c;\mathbb{Z}}(U) \right)$$

- Extension to $M \in \text{Loc}$ via homotopy colimit/homotopy left Kan extension

$$\mathcal{O}(M) := \text{hocolim} \left( \text{Loc}_\odot/M \xrightarrow{Q_M} \text{Loc}_\odot \xrightarrow{\mathcal{O}_\odot} \text{Ch}(\text{Ab}) \right)$$

**Theorem** [Benini,AS,Szabo]

For every $M \in \text{Loc}$, $\mathcal{O}(M)$ is weakly equivalent to dual Deligne complex on $M$.

In particular, it contains observables for flat connections and respects magnetic charge quantization of gauge theories!
Bird’s-eye view on homotopical AQFT

◊ **Higher structures in gauge theory:** [cf. Gwilliam’s talk]
  
  • ‘Spaces’ of gauge fields are not usual spaces, but higher spaces called **stacks**.
  
  • Consequently, observable ‘algebras’ for gauge theories are not conventional algebras, but higher algebras, e.g. **differential graded algebras**.

◊ To formalize **quantum gauge theories**, we develop

  \[
  \text{homotopical AQFT} := \text{AQFT} + \text{homotopical algebra}
  \]

‘Def.’ A homotopical AQFT is an assignment \( \mathcal{A} : \text{Loc} \to \text{dgAlg} \) of differential graded algebras (or other higher algebras) to spacetimes, satisfying

1. **functoriality**, **causality** and **time-slice** (possibly up to coherent homotopies);
2. **local-to-global property**, i.e. \( \mathcal{A} \) is homotopy left Kan extension of \( \mathcal{A}|_{\text{Loc}^\circ} \).

◊ **Our results:**

  • Global observables via homotopy left Kan extension [Benini,AS,Szabo]
  
  • Toy-models via orbifoldization (homotopy invariants) [Benini,AS]
  
  • Yang-Mills stack and stacky Cauchy problem [Benini,AS,Schreiber]
  
  • Towards a precise definition in terms of operads [Benini,AS,Woike]

**NB:** **BRST/BV formalism** of [Fredenhagen,Rejzner] should provide examples.
The operadic AQFT program

M. Benini, S. Bruinsma, AS and L. Woike
Categories of AQFTs: General perspective

- **Input data:** (so that we can talk about QFTs)
  - Category $C$ (‘spacetimes’) with subset $W \subseteq \text{Mor}C$ (‘Cauchy morphisms’)
  - Orthogonality $\perp \subseteq \text{Mor}C_t \times t\text{Mor}C$ (‘causally disjoint’ $(c_1 \to c \leftarrow c_2) \in \perp$)
  - Target category $M$ (bicomplete closed symmetric monoidal)

**Def:** The category of $M$-valued AQFTs on $(C, W, \perp)$ is the full subcategory $\text{qft}(C, W, \perp) \subseteq \text{Mon}_M^C$ of functors $\mathcal{A} : C \to \text{Mon}_M$ satisfying

1. **$W$-constancy:** For all $f \in W$, $\mathcal{A}(f)$ is isomorphism.
2. **$\perp$-commutativity:** For all $f_1 \perp f_2$, the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}(c_1) \otimes \mathcal{A}(c_2) & \xrightarrow{\mathcal{A}(f_1) \otimes \mathcal{A}(f_2)} & \mathcal{A}(c) \otimes \mathcal{A}(c) \\
\mathcal{A}(f_1) \otimes \mathcal{A}(f_2) \downarrow & & \downarrow \mu_c^\text{op} \\
\mathcal{A}(c) \otimes \mathcal{A}(c) & \xrightarrow{\mu_c} & \mathcal{A}(c)
\end{array}
\]

**Thm:** Localization induces equivalence $\text{qft}(C, W, \perp) \cong \text{qft}(C[W^{-1}], \emptyset, L_*(\perp))$.

! This means that $W$-constancy can be hard-coded as a structure.

**NB:** The relevant categories are $\text{QFT}(C, \perp) := \text{qft}(C, \emptyset, \perp)$. 
AQFTs are algebras over a colored operad

- **Operads** capture abstractly the operations underlying algebraic structures.
- **Example:** Associative and unital algebras

\[
\text{Ass-operad} \xrightarrow{\text{operad algebras}} \text{abstract algebras} \xrightarrow{\text{representations}} \text{concrete algebras}
\]

\[
\text{Ass}(n) = \{\mu_n\}, \text{ for } n \geq 0
\]

\[
\mu^n_A : A \otimes^n \to A
\]

\[
\mu_n : \text{End}(V) \otimes^n \to \text{End}(V)
\]

**Theorem** [Benini,AS,Woike]

For every \((\mathcal{C}, \perp)\), there exists an \(\text{Ob}(\mathcal{C})\)-colored operad \(\mathcal{O}_{(\mathcal{C}, \perp)}\) whose category of algebras is canonically isomorphic to the category of AQFTs on \((\mathcal{C}, \perp)\), i.e.

\[
\text{Alg}(\mathcal{O}_{(\mathcal{C}, \perp)}) \cong \text{QFT}(\mathcal{C}, \perp)
\]

- This means that \(\perp\)-commutativity can be hard-coded as a structure by using colored operads. ⇔ Very useful for universal constructions, see next slide.

**Rem:** Precise operadic definition of homotopical AQFT:

**homotopical AQFT** := \(\mathcal{O}_{(\mathcal{C}, \perp)}\) \(\infty\)-algebra + local-to-global property
A whole zoo of universal constructions

- **Main observation**: The assignment of AQFT operads \((C, \perp) \mapsto \mathcal{O}(C, \perp)\) is functorial \(\mathcal{O} : \text{OrthCat} \to \text{Op}(\mathcal{M})\) on the category of orthogonal categories.

\[ \Rightarrow \] For every orthogonal functor \(F : (C, \perp_C) \to (D, \perp_D)\) we obtain adjunction

\[ QFT(C, \perp_C) \congarr_{F^*} F! \rightarr QFT(D, \perp_D) \]

Because the \(W\)-constancy and \(\perp\)-commutativity axioms are hard-coded as structures in our operads, these adjunctions always produce AQFTs.

- **Examples**:

1. **\(\perp\)-Abelianization**: \(\text{id}_C : (C, \emptyset) \to (C, \perp)\) induces

\[ \text{Mon}_M^C \congarr_{\text{Ab}} \text{Forget} \rightarr QFT(C, \perp) \]

2. **\(W\)-constantification**: \(L : (C, \perp) \to (C[W^{-1}], L^*(\perp))\) induces

\[ QFT(C, \perp) \congarr_{L^*} L! \rightarr QFT(C[W^{-1}], L^*(\perp)) \]

3. **Local-to-global**: Full orthogonal subcat \(j : (C, j^*(\perp)) \to (D, \perp)\) induces

\[ QFT(C, j^*(\perp)) \congarr_{j^*} j! \rightarr QFT(D, \perp) \]
Comparison to Fredenhagen’s universal algebra

- Fredenhagen’s universal algebra ignores the $\perp$-commutativity axiom.
- Comparison via diagram of adjunctions (square of right adjoints commutes)

\[
\begin{array}{ccc}
\text{QFT}(\text{Loc} \otimes, j^*(\perp)) & \xrightarrow{j_!} & \text{QFT}(\text{Loc}, \perp) \\
\text{Ab} & \xrightarrow{\text{Forget}} & \text{Ab} \\
\text{Mon}^{\text{Loc} \otimes}_M & \xleftarrow{\text{Lan}_j} & \text{Mon}^{\text{Loc}}_M \\
\text{(operadic)} & & \text{(Fredenhagen)}
\end{array}
\]

**Theorem** [Benini,AS,Woike]

1. Let $\mathcal{A} \in \text{QFT}(\text{Loc} \otimes, j^*(\perp))$ be such that $\text{Lan}_j \text{Forget} \mathcal{A}$ is $\perp$-commutative. Then $\text{Lan}_j \text{Forget} \mathcal{A} \cong \text{Forget} j_! \mathcal{A}$.

2. $\text{Lan}_j \text{Forget} \mathcal{A}$ is $\perp$-commutative over $M \in \text{Loc}$, for all $\mathcal{A}$, if and only if for all causally disjoint $f_1 : U_1 \to M \leftarrow U_2 : f_2$ with $U_1, U_2 \in \text{Loc} \otimes$ there exists

\[
\begin{array}{ccc}
M & \xrightarrow{f_1} & U_1 \\
\text{f_2} & \xleftarrow{U_2} & \text{f_2}
\end{array}
\]

**Rem:** Property in 2. violated for disconnected $M$. Open question: Connected $M$?
A characterization theorem for boundary AQFTs

M. Benini, C. Dappiaggi and AS (to appear soon)
Universal boundary extensions of AQFTs

- Spacetime \( M \) with timelike boundary
- \( \text{QFT}(\text{int} M) \) on causally compatible interior regions
- \( \text{QFT}(M) \) on all causally compatible regions

- **Universal boundary extension** (no choice of boundary conditions needed!)

\[
\begin{align*}
\text{QFT}(\text{int} M) & \xrightarrow{\text{ext}} \text{QFT}(M) \\
\text{QFT}(M) & \xleftarrow{\text{res}} \text{QFT}(\text{int} M)
\end{align*}
\]

- Given \( \mathcal{B} \in \text{QFT}(M) \), the counit of the adjunction provides comparison map

\[
\epsilon_{\mathcal{B}} : \text{ext res } \mathcal{B} \rightarrow \mathcal{B}
\]

**Theorem** [Benini,Dappiaggi,AS]

\[\epsilon_{\mathcal{B}} : \text{ext res } \mathcal{B} / \text{ker} \epsilon_{\mathcal{B}} \rightarrow \mathcal{B} \] is isomorphism **if and only if** \( \mathcal{B} \) is generated from the interior, i.e. every \( \mathcal{B}(V) \) is generated by \( \mathcal{B}(V_{\text{int}}) \), for all interior regions \( V_{\text{int}} \subseteq V \).

**Rem:** Implies that every such \( \mathcal{B} \) may be described by two independent data:

1. A theory on the interior \( \mathcal{A} \in \text{QFT}(\text{int} M) \)
2. An ideal \( \mathcal{I} \subseteq \text{ext } \mathcal{A} \) that vanishes on the interior \( \xleftarrow{\text{boundary conditions}} \).
Happy Birthday, Klaus!

Klaus secretly doing homotopy theory in the black forest!

(MFO Mini-Workshop: New interactions between homotopical algebra and quantum field theory)