BV and BFV formalism beyond perturbation theory

Based on joint works w/ Benini-Safronov [2104.14886]
and Benini-Ridham [2201.10225]

■ What is derived geometry?

- "Traditional" geometric frameworks, such as manifolds or schemes, are incapable to describe certain important geometric objects:

(i) Quotients by non-free group actions:

X/G is in general singular, e.g.

and it ignores in how many ways points get identified, e.g.

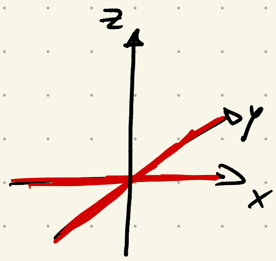
$*/G = *$ independently of G.

(ii) Non-transversal intersections:

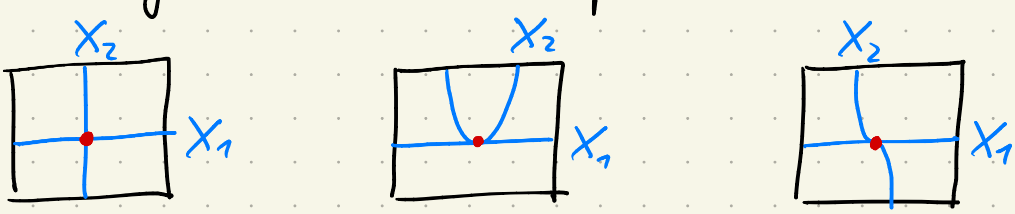
$X_1 \times_Y X_2$ is in general singular, e.g.

$Y=\mathbb{R}^3$, $X_1 = \{(x,y,z)\in\mathbb{R}^3 : z=0\}$ and $X_2 = \{(x,y,z)\in\mathbb{R}^3 : (xy)^2 = z\}$

$\rightsquigarrow X_1 \times_Y X_2 = \{(x,y,z)\in\mathbb{R}^3 : (xy)^2=0 \text{ and } z=0\}$



and it ignores intersection multiplicities



- Derived (algebraic) geometry resolves these issues by introducing a powerful and refined concept of space called <u>derived stacks</u>.

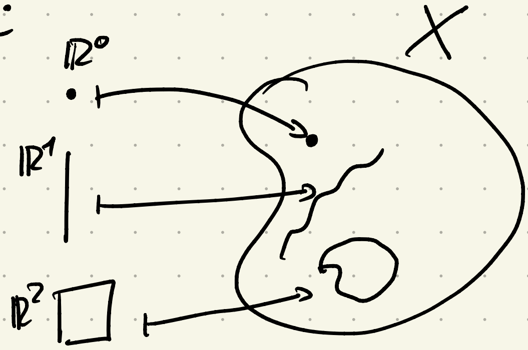
- To get some intuition, we have to recall <u>functors of points</u>:

$$\mathrm{Aff} := \mathrm{CAlg}^{op} \xrightarrow[\text{(fully faithful)}]{\text{Yoneda}} \mathrm{Sh}(\mathrm{Aff}, \mathrm{Set})$$

affine schemes (the building blocks) — schemes and more (what you get from glueing)

<u>Interpretation of a functor $X : \mathrm{Aff}^{op} \to \mathrm{Set}$:</u>

- $X(\mathbb{R}^0)$ = "points in X"
- $X(\mathbb{R}^1)$ = "curves in X"
- $X(\mathbb{R}^2)$ = "surfaces in X"
- ...



- Derived stacks have a richer functor of points:

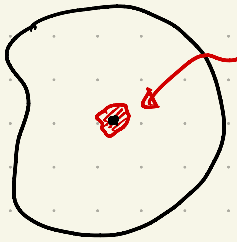
improves intersections ⇝ EOM or contraints (antifields)

$$\begin{tikzcd}
\mathrm{Aff}^{op} = \mathrm{CAlg} \arrow[rr, "\text{scheme}"] \arrow[rrd, "\text{stack}"] \arrow[rrddd, "\infty\text{-stack}"] \arrow[ddd, hook] & & \mathrm{Set} \arrow[d, hook] \\
& & \mathrm{Grpd} \arrow[d, hook] \\
& & \vdots \arrow[d, hook] \\
\mathrm{dAff}^{op} = \mathrm{dgCAlg}_{\geq 0} \arrow[rr, "\text{derived } (\infty\text{-})\text{stack}"'] & & \infty\mathrm{Grpd}
\end{tikzcd}$$

improves quotients ⇝ gauge symmetry (ghosts)

! $\mathrm{dSt} := \mathrm{Sh}_\infty(\mathrm{dAff}, \infty\mathrm{Grpd})$ defines an <u>∞-topos</u>, hence all quotients, intersections, etc., <u>exist</u>. The challenge for applications is therefore to describe your derived stacks of interest in an explicit and useful way...

- How is all of this related to BRST, BV, BFV, ...?



formal neighborhood around point $\overset{\text{Lurie}}{\underset{\text{Pridham}}{\Longleftrightarrow}}$ L_∞-algebra (or DGCAlg)

$X \in dSt$

$\rightsquigarrow$ $\boxed{\text{BRST/BV/BFV/}\cdots = \text{perturbative/formal aspects of derived geometry}}$

Application 1: Non-perturbative (classical and fin. dim.) BV formalism

- Physical scenario:

A physical system is typically described by

1.) a space of fields X,

2.) an action of the gauge group $X \times G \longrightarrow X$,

3.) a gauge-invariant action function S

Want to determine the space of extrema of S modulo gauge, which involves taking quotients (X/G) and intersections $(d^{dR} S = 0)$ $\rightsquigarrow$ DAG potentially important!

- Mathematical formalization:

Let $X = \mathrm{Spec}\, A$ be smooth affine scheme w/ action $X \times G \longrightarrow X$ of a smooth affine group scheme $G = \mathrm{Spec}\, H$. Consider quotient stack

$$[X/G] := \mathrm{colim}\Big(X \leftleftarrows X \times G \overset{\leftarrow}{\leftleftarrows} X \times G^2 \cdots \Big) \in dSt$$

and a function $S : [X/G] \longrightarrow \mathbb{R}$.

The derived critical locus is defined as the pullback in dSt

$$\begin{tikzcd} \mathrm{dCrit}(S) \arrow[r, dashed] \arrow[d, dashed] & {[X/G]} \arrow[d, "0"] \\ {[X/G]} \arrow[r, "d^{dR}S"'] & T^*[X/G] \end{tikzcd}$$

OK, that's a bit abstract and inexplicit...

Can one understand better how dCrit(S) "looks like" and how it relates to the physicists BV formalism?

• Thm: (Benini–Safronov–AS)

$\mathrm{dCrit}(S:[X/G]\to\mathbb{R}) \simeq [Z/G]$ is a derived quotient stack, where $Z = \mathrm{Spec}\,\mathcal{O}(Z)_\bullet$ is derived affine scheme with function dg-algebra determined by the two homotopy pushouts in $\mathrm{dgCAlg}_{\geq 0}$

$$\begin{tikzcd} \mathrm{Sym}\,\mathfrak{g} \arrow[r, "0^*"] \arrow[d, "\mu^*"'] & \mathbb{R} \arrow[d, dashed] \\ \mathrm{Sym}_A T_A \arrow[r, dashed] \arrow[ur, phantom, "h"] & \mathcal{O}(\mu^{-1}(0))_\bullet \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(\mu^{-1}(0))_\bullet \arrow[r, "0^*"] \arrow[d, "(d^{dR}S)^*"'] & A \arrow[d, dashed] \\ A \arrow[r, dashed] \arrow[ur, phantom, "h"] & \mathcal{O}(Z)_\bullet \end{tikzcd}$$

An explicit model is given by

$$\mathcal{O}(Z)_\bullet = \mathrm{Sym}_{\underbrace{A}_{\text{fields}}}\Big(\underbrace{A\otimes\mathfrak{g}_{[-2]}}_{\substack{\text{antifields}\\\text{for ghosts}}} \oplus \underbrace{T_{A\,[-1]}}_{\text{antifields}}\Big) \in \mathrm{dgCAlg}_{\geq 0}$$

with differential

tautological 1-form on T^*X

$$\partial a = 0 \quad , \quad \partial v = i_v\big(d^{dR}S\big) \quad , \quad \partial t = \mu^*(t) = -\,i_{\rho(t)}\lambda$$

• From this perspective, it is easy to determine the function dg-algebra

$$\mathcal{O}\big(\mathrm{dCrit}(S)\big) \simeq \mathcal{O}\big([Z/G]\big)$$
$$\simeq \mathcal{O}\Big(\mathrm{colim}\big(Z \leftleftarrows Z\times G \,\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}\, Z\times G^2 \cdots\big)\Big)$$

$$\simeq \lim \left(\mathcal{O}(Z)_\bullet \rightrightarrows \mathcal{O}(Z\times G)_\bullet \substack{\to\\\to\\\to} \mathcal{O}(Z\times G^2)_\bullet \cdots \right)$$

$$\simeq \mathrm{Tot}^{\Pi} \underbrace{N^\bullet(G, \mathcal{O}(Z)_\bullet)}_{\substack{\text{normalized group cochains}\\ \text{w/ coefficients in } \mathcal{O}(Z)_\bullet}} \in \mathrm{DGCAlg}$$

The BV formalism considers instead a Lie algebra quotient

$$BV(S) = [Z/\mathfrak{g}] \Rightarrow \mathcal{O}(BV(S)) \simeq \mathrm{Tot}^{\Pi}\, CE^\bullet(\mathfrak{g}, \mathcal{O}(Z)_\bullet) \in \mathrm{DGCAlg}$$

These two are related by the van Est map $vE : \mathcal{O}(d\mathrm{Crit}(S)) \xrightarrow{\substack{\text{i.g. not}\\ \text{equivalence!}}} \mathcal{O}(BV(S))$, which further preserves the canonical (-1)-shifted symplectic structures.

- Warning: $d\mathrm{Crit}(S) \simeq [Z/G]$ is in general NOT affine, i.e. $\mathcal{O}(d\mathrm{Crit}(S))$ does NOT faithfully encode this derived stack.

  This is a new feature of the non-perturbative world!

  It is however faithfully encoded by its dg-category of modules

  $$\mathrm{QCoh}(d\mathrm{Crit}(S)) \simeq G\text{-}\mathrm{dgMod}_{\mathcal{O}(Z)_\bullet}$$

■ Application 2: Quantization of gauge-theoretic canonical phase spaces

- The canonical phase space of 2^{nd}-order gauge theories is a derived cotangent stack

$$\underbrace{T^*}_{\substack{\text{canonical}\\ \text{momenta}}}[\underbrace{X}_{\substack{\text{position}\\ \text{variables}}}/\underbrace{G}_{\substack{\text{gauge}\\ \text{symmetries}}}] \overset{\text{Safronov}}{\simeq} \underbrace{[T^*X/\!/G]}_{\substack{\text{Symplectic}\\ \text{reduction}}} := [\mu^{-1}(0)/G]$$

where (as before) $\mu^{-1}(0) = \mathrm{Spec}\, \mathcal{O}(\mu^{-1}(0))_\bullet$ is derived affine scheme with

$$\mathcal{O}(\mu^{-1}(0))_\bullet = \mathrm{Sym}_A \left(T_A \xleftarrow{\mu^*} A \otimes \mathfrak{g}_{[-1]} \right) \in \mathrm{dgCAlg}_{\geq 0}$$

- Studying the Poisson geometry and quantization of $T^*[X/G]$ one faces two main challenges:
  1.) Poisson structures are poorly functorial, so a Poisson structure on
  $$T^*[X/G] \simeq [\mu^{-1}(0)/G] = \operatorname{colim}\left(\mu^{-1}(0) \leftleftarrows \mu^{-1}(0)\times G \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \cdots\right)$$
  is <u>NOT</u> simply a family of compatible Poisson structures on each level.

  2.) $T^*[X/G]$ is <u>NOT</u> affine, so we have to quantize the dg-category $QCoh(T^*[X/G]) \simeq G\text{-}dgMod_{\mathcal{O}(\mu^{-1})}$.

- At the abstract level, these challenges have been solved by CPTVV and independently by Pridham. Can one make this explicit?
  ⇝ That's the content of the Benini-Pridham-AS paper.

- Even though the technical details are tough, the <u>KEY IDEA</u> is simple and beautiful:

<u>Step 1:</u> Resolve $T^*[X/G] \simeq [\mu^{-1}(0)/G]$ by a diagram of Lie algebra quotients

that's the BFV formalism

$$[\mu^{-1}(0)/\mathfrak{g}] \leftleftarrows [\mu^{-1}(0)\times G/\mathfrak{g}\oplus\mathfrak{g}] \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \cdots$$

This is level-wise affine and we can pass to stacky dg-algebras

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet) \rightrightarrows CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet) \;\overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

The face and degeneracy maps are formally étale, so a Poisson structure can be defined level wise. At level 0, the non-trivial Poisson brackets read as

$$\{v, a\} = v(a) \quad , \quad \{v, v'\} = [v, v'] \, , \quad \{t, \theta\} = -\langle \theta, t\rangle$$

<u>Step 2:</u> Quantize level wise in terms of differential operators

$$\widehat{CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)}_\hbar \rightrightarrows \widehat{CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet)}_\hbar \;\overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

and pass over to dg-categories of modules

$$\circledast = \Big(\mathrm{dgMod}_{\widehat{CE^\bullet(\mathfrak{g},\mathcal{O}(\mu^{-1}(0)))_\bullet}{}_\hbar} \rightrightarrows \mathrm{dgMod}_{\widehat{CE^\bullet(\mathfrak{g}\oplus\mathfrak{g},\mathcal{O}(\mu^{-1}(0)\times G))_\bullet}{}_\hbar} \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \cdots \Big)$$

Step 3: Obtain global quantization of $T^*[X/G]$ by computing homotopy limit (in dgCat) of the local quantizations

$$\mathrm{QCoh}(T^*[X/G])_\hbar := \mathrm{holim}\ \circledast \in \mathrm{dgCat}$$

- Thm: (Benini - Pridham - AS)

A model for the dg-category $\mathrm{QCoh}(T^*[X/G])_\hbar$ is given by the following:

Objects: Triples $(\mathcal{E}_\bullet, \nabla, \Psi)$ consisting of

(1) an $A[[\hbar]]$-dg-module $\mathcal{E}_\bullet$ w/ G-action (wave functions w/ G-action)

(2) a G-eqv. dg-connection $\nabla : \mathcal{E}_\bullet \longrightarrow \Omega^1_A[[\hbar]] \otimes_{A[[\hbar]]} \mathcal{E}_\bullet$ w.r.t. $\hbar d^{dR}$, i.e. $\underbrace{\nabla(a\,s) = \hbar\, d^{dR}a \otimes s + a.\nabla(s)}_{\text{CCR position variables and momenta}}$ (action of canonical momenta)

(3) a G-eqv. graded module map $\Psi : \mathfrak{g}_{[-1]} \otimes \mathcal{E}_\# \longrightarrow \mathcal{E}_\#$ (action of antighosts)

satisfying

(i) $\underbrace{\nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar \nabla_{[v,v']}\,, \quad \nabla_v \Psi_t = \Psi_t \nabla_v\,, \quad \Psi_t \Psi_{t'} = -\Psi_{t'} \Psi_t}_{\text{CCR momenta and antighosts}}$

(ii) $\underbrace{\partial \Psi_t + \Psi_t \partial = \nabla_{\mu^*(t)} + \hbar\, \rho(t)}_{\text{CCR ghosts and antighosts}}$

(+ technical conditions)

Morphisms: $\underline{\mathrm{hom}}\big((\mathcal{E}_\bullet, \nabla, \Psi), (\mathcal{E}'_\bullet, \nabla', \Psi')\big) := \underline{\mathrm{hom}}_A^{G,\nabla,\Psi}(\mathcal{E}_\bullet, \mathcal{E}'_\bullet) \subseteq \underline{\mathrm{hom}}_A(\mathcal{E}_\bullet, \mathcal{E}'_\bullet)$