Boundary Conditions and Edge Modes in Gauge Theories

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Based joint works w/ P. Mathieu, L. Murray and N. Teh [1907.10651]
and w/ M. Benini and B. Vicedo [2008.01829]

Plan of the talk:

1.) Stacky point of view on boundary conditions ⇝ "Edge modes"

2.) Illustration through the Abelian Chern-Simons / WZW system

3.) Application to the 4d holomorphic Chern-Simons / 2d integrable field theory correspondence by Costello-Yamazaki-Witten

Boundary conditions in "ordinary" field theories

- Consider for concreteness a scalar field $\Phi \in C^\infty(M)$ on a manifold M w/ boundary $\partial M \neq \emptyset$.
- A boundary condition is a condition on the value of the field Φ (or more generally its jets) on ∂M.

  Example: $\Phi|_{\partial M} = 0$ in $C^\infty(\partial M)$ [Dirichlet boundary condition]

- From a categorical point of view, enforcing a boundary condition can be understood as computing a pullback in a suitable category of spaces:

$$\begin{tikzcd}
\boxed{\text{Fields w/ boundary condition}} \arrow[r, dashed] \arrow[d, dashed] & \text{Bulk Fields} \arrow[d] \\
\text{Bdy Fields} \arrow[r] & \text{Comparison}
\end{tikzcd}$$

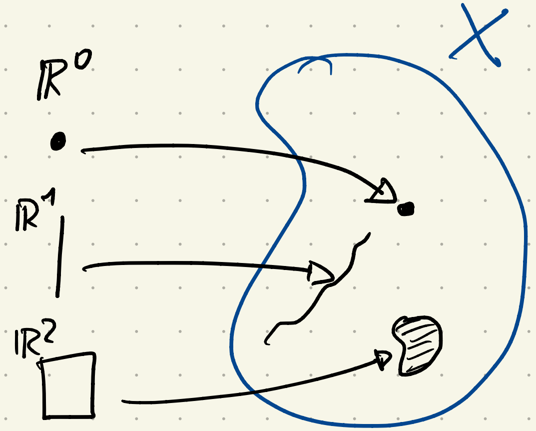
Example:

$$\begin{tikzcd}
C^\infty_{\text{Dirichlet}}(M) \arrow[r, dashed] \arrow[d, dashed] & C^\infty(M) \arrow[d, "\text{res}"] \\
\{*\} \arrow[r, "0"'] & C^\infty(\partial M)
\end{tikzcd}$$

Quick recap of stacks in gauge theory

- A **stack** is a (pseudo-)functor $X : \mathrm{Man}^{op} \longrightarrow \mathrm{Grpd}$ satisfying a 2-categorical descent condition for open covers.

This is interpreted as a functor of points:



- $X(\mathbb{R}^0)$ = groupoid of points of X
- $X(\mathbb{R}^1)$ = groupoid of smooth curves in X
- $X(\mathbb{R}^2)$ = groupoid of smooth planes in X ...

- Example: Stack of gauge fields w/ group G on Cartesian space $U \cong \mathbb{R}^m$

$$\mathrm{Con}_G(U) : \mathrm{Man}^{op} \longrightarrow \mathrm{Grpd}$$

$$N \longmapsto \mathrm{Con}_G(U)(N) = \underbrace{\left\{ \begin{array}{ll} \underline{\text{Obj:}} & A \in \Omega^{1,0}(U \times N, \mathfrak{g}) \\ \underline{\text{Mor:}} & A \xrightarrow{g} g^{-1}Ag + g^{-1}d_U g \\ & \text{with } g \in C^\infty(U \times N, G) \end{array} \right.}_{\text{smoothly } N\text{-parametrized gauge fields and transformations on } U.}$$

Stacky point of view on boundary conditions

- **Key point:** Stacks form a 2-category $\rightsquigarrow$ 2-categorical pullbacks
- The 2-pullback implementing a boundary condition is of the form

$$\begin{tikzcd} \mathcal{F} \arrow[r, dashed] \arrow[d, dashed, "\text{T2-pullback}"] & \mathrm{BulkFields} \arrow[d, "\mathrm{res}"] \\ \mathrm{BdyFields} \arrow[r, "i"'] & \mathrm{Comparison} \end{tikzcd}$$

$$\mathcal{F}(N) = \begin{cases} \underline{\text{Obj:}} & (A, B, \operatorname{res} A \xrightarrow{g} iB) \\ \underline{\text{Mor:}} & (\phi, \psi) : (A, B, g) \longrightarrow (A', B', g') \\ \text{s.t.} & \begin{tikzcd} \operatorname{res} A \arrow[r, "\operatorname{res}\phi"] \arrow[d, "g"'] & \operatorname{res} A' \arrow[d, "g'"] \\ iB \arrow[r, "i\psi"'] & iB' \end{tikzcd} \end{cases}$$

- **Example:** Gauge fields w/ group $\mathbb{R}$ and boundary condition

"restriction of bulk bundle = fixed bundle on ∂M"

$$\begin{tikzcd} \mathcal{F} \arrow[r, dashed] \arrow[d, dashed, "\text{T2-pullback}"] & \mathrm{Con}_{\mathbb{R}}(M) \arrow[d, "\text{restrict bundle}"] \\ \{*\} \arrow[r, "\text{pick bundle}"'] & \mathrm{Bun}_{\mathbb{R}}(\partial M) \end{tikzcd}$$

edge modes

$$\mathcal{F}(N) \simeq \begin{cases} \underline{\text{Obj:}} & (A, \varphi) \in \Omega^{1,0}(M \times N) \times \Omega^0(\partial M \times N) \\ \underline{\text{Mor:}} & (A, \varphi) \xrightarrow{\varepsilon} (A + d_M \varepsilon, \varphi + \varepsilon|_{\partial M}) \\ & \text{with } \varepsilon \in \Omega^0(M \times N) \end{cases}$$

Abelian Chern-Simons/WZW system

- Let M be a 3-mnf w/ boundary ∂M and consider gauge fields w/ group $\mathbb{R}$ subject to boundary condition "restriction of bulk bundle = fixed bundle on ∂M".
- On the previous slide we have seen that the stack of fields is

$$\mathcal{F}(N) \simeq \begin{cases} \underline{\text{Obj:}} \quad (A, \varphi) \in \Omega^{1,0}(M\times N) \times \Omega^0(\partial M \times N) \\ \underline{\text{Mor:}} \quad (A, \varphi) \xrightarrow{\varepsilon} (A + d_M \varepsilon, \varphi + \varepsilon|_{\partial M}) \quad \text{with } \varepsilon \in \Omega^0(M\times N) \end{cases}$$

- Choosing a Lorentzian metric on ∂M, we can define action $S: \mathcal{F} \longrightarrow \mathbb{R}$

$$S(A,\varphi) = \underbrace{\int_M \tfrac{1}{2} A \wedge d_M A}_{\text{bulk CS action}} + \underbrace{\int_{\partial M} \tfrac{1}{2}\Big(d_{\partial M}\varphi \wedge A|_{\partial M} + \lambda \, \overbrace{d_A \varphi}^{= d_{\partial M}\varphi - A|_{\partial M}} \wedge * d_A \varphi \Big)}_{\text{boundary action}}$$

Euler-Lagrange equations $\rightsquigarrow$

$$\begin{cases} d_M A = 0 & \text{on } M \leftarrow \text{flat connections in bulk} \\ d_A \varphi + 2\lambda * d_A \varphi = 0 & \text{on } \partial M \leftarrow \text{for } \lambda = \pm\tfrac{1}{2}\text{, (anti) chiral current } d_A\varphi \text{ on boundary} \end{cases}$$

Application to the 4d Chern-Simons / 2d integrable field theory correspondence

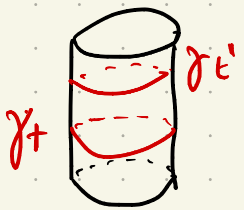
- A field theory on a 2d spacetime Σ is classically integrable if its equations of motion are of Lax form:

$$\boxed{EOM = 0} \iff \boxed{d_\Sigma \mathcal{L}(z) + \tfrac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = 0}$$

flatness

where $\mathcal{L} \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$ is connection depending meromorphically on some Riemann surface C.

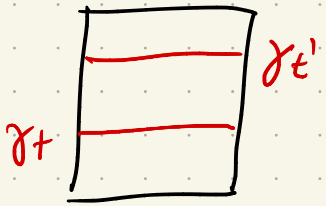
- The Lax connection $\mathcal{L}(z)$ defines ∞-many conserved charges through holonomy:



γ_t $\gamma_{t'}$

$\Sigma = \mathbb{R} \times S^1$

or



γ_t $\gamma_{t'}$

$\Sigma = \mathbb{R}^2$

$$\rightsquigarrow \quad \partial_t \, \mathrm{tr}\left[P\exp \int_{\gamma_t} \mathcal{L}(z) \right] = 0$$

because $\mathcal{L}$ flat

leads to ∞-many conserved charges by expanding in z

- Main problem in IFT:

Where does the Lax connection come from?

Geometric approach via 4d holomorphic Chern-Simons theory

- Costello-Yamazaki-Witten proposed a geometric approach to construct 2d IFTs on $\Sigma = \mathbb{R}^2$ from a certain gauge theory on 4-mnf $X := \Sigma \times C'$. ($C' = \mathbb{CP}^1 \setminus \xi$)
- The input for their construction is:
  (i) Lie group G w/ invariant inner product $\langle \cdot , \cdot \rangle$ on Lie algebra $\mathfrak{g}$

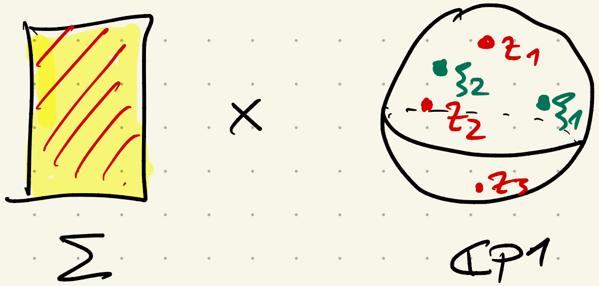
  (ii) Fixed meromorphic 1-form ω on $\mathbb{CP}^1$, with poles denoted by (z_i) and zeros denoted by $\xi = (\xi_a)$ → remove zeros of ω

  (iii) The following action for connections $A \in \Omega^1(X, \mathfrak{g})$

$$\boxed{S_\omega(A) = \frac{i}{4\pi} \int_X \omega \wedge CS(A)}$$

with $CS(A) := \left\langle A, dA + \frac{1}{3}[A, A] \right\rangle \in \Omega^3(X)$ the Chern-Simons 3-form.

  (iv) Boundary conditions for A on surface defect $D := \bigsqcup_i \Sigma \times \{z_i\}$



Details on boundary conditions I

- For simplicity, I consider here only the case of simple poles $\omega = \sum_i \frac{k_i}{z - z_i} dz$. ↙ residues
- Keypoint: Gauge invariance of action S_ω requires boundary conditions!

For $A \xrightarrow{g \in C^\infty(X,G)} A^g = g^{-1} A g + g^{-1} dg$, we have

S_ω is NOT gauge invariant

$$S_\omega(A^g) = S_\omega(A) \boxed{- \frac{i}{4\pi} \int_X \omega \wedge d\langle g^* \Theta_G, A\rangle - \frac{i}{4\pi} \int_X \omega \wedge g^* \chi_G}$$

with $\Theta_G \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form and $\chi_G = \frac{1}{6}\langle \Theta_G, [\Theta_G, \Theta_G]\rangle \in \Omega^3(G)$.

- Introducing the defect group $G^{\underline{z}} := \prod_{\text{poles}} G$ and the inner product $\langle\langle X, Y\rangle\rangle_\omega := \sum_i k_i \langle X_i, Y_i\rangle$ on its Lie algebra $\mathfrak{g}^{\underline{z}} = \prod_{\text{poles}} \mathfrak{g}$, we can prove:

- Theorem (BSV):

$$S_\omega(A^g) = S_\omega(A) + \frac{1}{2} \int_\Sigma \langle\langle (i^* g)^* \Theta_{G^{\underline{z}}}, i^* A\rangle\rangle_\omega - \frac{1}{2} \int_{\Sigma \times (0,1)} \hat{g}^* \chi_{G^{\underline{z}}}$$

where $i : D \hookrightarrow X$ is defect inclusion and $\hat{g}$ is any lazy homotopy from $i^* g$ to e.

Details on boundary conditions II

- Generalizing Costello et al., we consider boundary conditions of the form

$$i^*A \in \Omega^1(\Sigma, \mathfrak{k}) \quad \underline{\text{and}} \quad i^*g \in C^\infty(\Sigma, \mathbb{K})$$

where $\mathfrak{k} \subseteq \mathfrak{g}^{\mathbb{Z}}$ is isotropic sub-Lie algebra and $\mathbb{K} \subseteq G^{\mathbb{Z}}$ its Lie integration.

- Corollary: Enforcing these boundary conditions in the strict sense defines a sub-stack $\mathcal{F}_{bc}(X) \overset{\subseteq}{\hookrightarrow} \mathrm{Con}_G(X)$ on which $S_w : \mathcal{F}_{bc}(X) \longrightarrow \mathbb{R}$ defines a gauge invariant action.

- Remark: Enforcing the boundary conditions via a $\mathbb{Z}$-pull back defines another stack $\mathcal{F}(X)$ together with a morphism $\mathcal{F}_{bc}(X) \longrightarrow \mathcal{F}(X)$.

One can prove that this morphism is an equivalence of stacks, hence the strict and the higher categorical point of view on boundary conditions is equivalent in this case!

Where does the 2dIFT come from?

- There exists a zig-zag of stack morphisms:

$$\begin{tikzcd}
\underbrace{\mathcal{F}_{bc}(X)}_{\substack{\text{strict bdy}\\ \text{conditions}}} \arrow[r, "\sim"] & \underbrace{\mathcal{F}(X)}_{\substack{\text{"weak" bdy}\\ \text{conditions}\\ \rightsquigarrow \text{ edge modes}}} & \underbrace{\mathcal{F}_{Lax}(X)}_{\substack{\text{Lax-type}\\ \text{connections}\\ + \text{ edge modes}}} \arrow[l, hook', "\subseteq"'] \arrow[r, "\pi"] & \underbrace{\mathcal{F}_{2d}(\Sigma)}_{\substack{\text{2d fields}\\ \hat{=} \text{ edge modes}}} \arrow[l, dashed, bend left, "s"]
\end{tikzcd}$$

- CHOOSING a section $s: \mathcal{F}_{2d}(\Sigma) \longrightarrow \mathcal{F}_{Lax}(X)$ of π, we can transfer the action S_ω on $\mathcal{F}_{bc}(X)$ to an action S_{2d} on $\mathcal{F}_{2d}(\Sigma)$.

Writing $s(h) = (\mathcal{L}(h), h)$, for $h \in C^\infty(\Sigma, G^\mathbb{Z})$ edge mode, we get

$$\boxed{S_{2d}(h) = -\frac{1}{2}\int_\Sigma \langle\!\langle h^*\Theta_{G^\mathbb{Z}}, i^*\mathcal{L}(h)\rangle\!\rangle_\omega + \frac{1}{2}\int_{\Sigma\times(0,1)} \hat{h}^*\chi_{G^\mathbb{Z}}}$$

lazy homotopy from h to e.

$\rightsquigarrow$ ACTIONS FOR 2dIFTs !

End