# Quantization of derived cotangent stacks

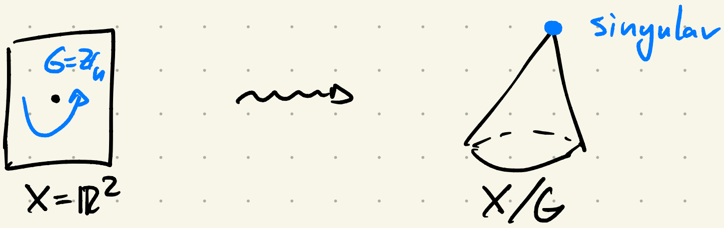
Joint work w/ M. Benini and J.P. Pridham [2201.10225]

## What is derived geometry?

- "Traditional" geometric frameworks, such as manifolds or schemes, are incapable to describe important geometric situations:

(i) Quotients by non-free group actions:

$X/G$ is in general singular, e.g.




and ignores in how many ways points get identified, e.g.

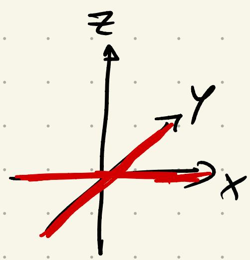
$*/G = *$ independently of $G$.

(ii) Non-transversal intersections:

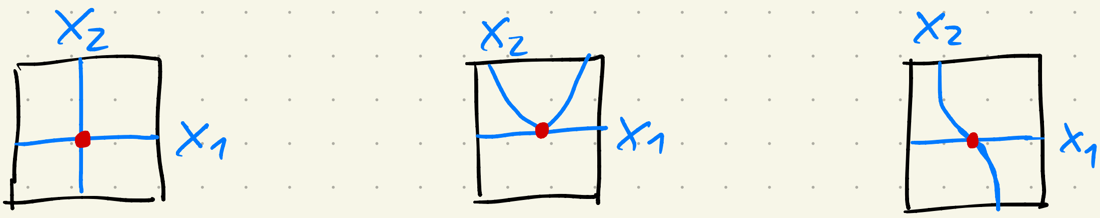
$X_1 \underset{Y}{\times} X_2$ is in general singular, e.g.

$Y=\mathbb{R}^3$, $X_1 = \{(x,y,z) \in \mathbb{R}^3 : z=0\}$, $X_2 = \{(x,y,z) \in \mathbb{R}^3 : (xy)^2 = z\}$

$\rightsquigarrow X_1 \underset{Y}{\times} X_2 = \{(x,y,z) \in \mathbb{R}^3 : (xy)^2 = 0 \text{ and } z=0\}$




and it ignores intersection multiplicities, e.g.



- Derived (algebraic) geometry resolves these issues by introducing a refined concept of space called <u>derived stacks</u>.

- To get some intuition, we have to recall <u>functors of points</u>:

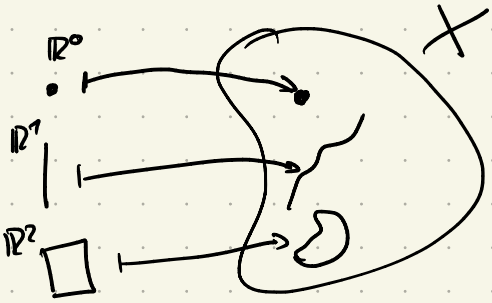
$$\mathrm{Aff} := \mathrm{CAlg}^{op} \xhookrightarrow[\text{(fully faithful)}]{\text{Yoneda}} \mathrm{Sh}(\mathrm{Aff}, \mathrm{Set})$$

affine schemes
(the building blocks)

schemes and more
(what you get from glueing)

Interpretation of a functor $X : \mathrm{Aff}^{op} \longrightarrow \mathrm{Set}$:

- $X(\mathbb{R}^0)$ = "points in $X$"
- $X(\mathbb{R}^1)$ = "curves in $X$"
- $X(\mathbb{R}^2)$ = "surfaces in $X$" ...




- Derived stacks have a richer functor of points:

improves intersections

$$\begin{tikzcd}
\mathrm{Aff}^{op} = \mathrm{CAlg} \arrow[rr, "\text{scheme}"] \arrow[rrd, "\text{stack}"] \arrow[rrdd, "\infty\text{-stack}"] \arrow[dd, hook] & & \mathrm{Set} \arrow[d, hook] \\
& & \mathrm{Grpd} \arrow[d, hook, "\vdots"] \\
\mathrm{dAff}^{op} = \mathrm{dgCAlg}_{\geq 0} \arrow[rr, "\text{derived } (\infty\text{-})\text{stack}"'] & & \infty\mathrm{Grpd}
\end{tikzcd}$$

improves quotients

**!** $\mathrm{dSt} := \mathrm{Sh}_\infty(\mathrm{dAff}, \infty\mathrm{Grpd})$ defines an <u>$\infty$-topos</u>, hence all quotients, intersections, etc., exist. The challenge for applications is therefore to describe your derived stack of interest in an explicit and useful way.

## Shifted Poisson structures and quantization:

- For a derived Artin stack $X$, there is a concept of Poisson structures and their quantization [Calaque, Pantev, Toen, Vaquie, Vezzosi / Pridham]

New features:

(i) These structures may have nontrivial cohomological degree $n \in \mathbb{Z}$!

(ii) $X$ is often NOT affine!

$\rightsquigarrow$ quantization of $\mathcal{O}(X) \in \mathrm{dgCAlg}$ is insufficient,

consider instead quantization of $\mathrm{QCoh}(X) \in \mathrm{SMdgCat}$

(iii) The shift $n \in \mathbb{Z}$ determines the type of quantization!

$$\big(X \text{ with } n\text{-shifted Poisson structure}\big)$$
$$\Big\downarrow \text{quantize}$$
$$\big(\mathrm{QCoh}(X)_\hbar \text{ is } E_n\text{-monoidal dg-category}\big)$$

- Example: (Safronov)

Consider $BG := [*/G] \in \mathrm{dSt}$ for a (reductive) affine group scheme $G$.

Then $\mathcal{O}(BG) \simeq C^\bullet(G, \mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ is not interesting, but

$\mathrm{QCoh}(BG) \simeq \mathrm{dgRep}(G)$ is interesting.

An $n=2$ shifted Poisson structure on $BG$ is an element $\pi \in (\mathrm{Sym}^2 \mathfrak{g})^G$.

The associated quantization $\mathrm{QCoh}(BG)_\hbar$ is the $E_2 =$ braided-monoidal dg-representation category of the associated quantum group.

■ Our contribution: (Quantizing derived cotangent stacks)

- Canonical phase space of $2^{nd}$-order gauge theories:

$$\underbrace{T^*}_{\text{canonical momenta}} [\underbrace{X}_{\text{position variables}} / \underbrace{G}_{\text{gauge symmetries}}] \overset{\text{Safronov}}{\simeq} \underbrace{[T^*X /\!\!/ G]}_{\text{symplectic reduction}} := [\mu^{-1}(0)/G]$$

- Here $\mu^{-1}(0) = \operatorname{Spec} \mathcal{O}(\mu^{-1}(0))_\bullet$ is the derived affine scheme describing the derived 0-locus of the moment map:

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, dashed] \arrow[d, dashed] & * \arrow[d, "0"] \\ T^*X \arrow[r, "\mu"'] & \mathfrak{g}^\vee \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} \operatorname{Sym}\mathfrak{g} \arrow[r, "0^*"] \arrow[d, "\mu^*"'] & \mathbb{K} \arrow[d, dashed] \\ \operatorname{Sym}_A T_A \arrow[r, dashed] & \mathcal{O}(\mu^{-1}(0))_\bullet \end{tikzcd}$$

$\infty$-pullback in dSt — homotopy pushout in $dgCAlg_{\geq 0}$

Explicit model:

$$\mathcal{O}(\mu^{-1}(0))_\bullet = \operatorname{Sym}_A \left( T_A \xleftarrow{\mu^*} A \otimes \mathfrak{g}[-1] \right) \in dgCAlg_{\geq 0}$$

- Wanted:

Quatization of $T^*[X/G]$ along the canonical 0-shifted Poisson structure!

How does the resulting $E_0$ = pointed dg-category look like:

$$QCoh(T^*[X/G])_\hbar = \, ?$$

- **Strategy**: Turn Pridham's abstract deformation theoretic arguments into a concrete construction! Let me sketch the KEY IDEAS:

Step 1: Resolve $T^*[X/G] \simeq [\mu^{-1}(0)/G]$ by a diagram of Lie algebra quotients:

$$\left[\mu^{-1}(0)/\mathfrak{g}\right] \leftleftarrows \left[\mu^{-1}(0)\times G/\mathfrak{g}\rtimes\mathfrak{g}\right] \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \cdots$$

This turns the global problem into a family of local stacky affine problems:

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet) \rightrightarrows CE^\bullet(\mathfrak{g}\rtimes\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

Step 2: Quantize level wise in terms of differential operators

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)_\hbar \rightrightarrows CE^\bullet(\mathfrak{g}\rtimes\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet)_\hbar \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

and pass over to dg-categories of modules:

$$\circledast := \left(\mathrm{dgMod}_{CE^\bullet(\mathfrak{g},\mathcal{O}(\mu^{-1}(0))_\bullet)_\hbar} \rightrightarrows \mathrm{dgMod}_{CE^\bullet(\mathfrak{g}\rtimes\mathfrak{g},\mathcal{O}(\mu^{-1}(0)\times G)_\bullet)_\hbar} \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots\right)$$

Step 3: Obtain global quantization of $T^*[X/G]$ by computing the homotopy limit (in dgCat) of the local quantizations:

$$\mathrm{QCoh}(T^*[X/G])_\hbar := \mathrm{holim}\, \circledast \in \mathrm{dgCat}$$

- **Thm:** (Benini–Pridham–AS)

A model for $\mathrm{QCoh}(T^*[X/G])_\hbar$ is given by the following:

**Objects:** Triples $(\mathcal{E}_\bullet, \nabla, \psi)$ consisting of

(1) an $\mathcal{O}(X)[[\hbar]]$-dg-module $\mathcal{E}_\bullet$ w/ $G$-action (wave functions w/ $G$-action)

(2) a $G$-eqv. dg-connection $\nabla : \mathcal{E}_\bullet \longrightarrow \Omega^1(X)[[\hbar]] \otimes_{\mathcal{O}(X)[[\hbar]]} \mathcal{E}_\bullet$ (action of canonical momenta)

w.r.t. $\hbar\, d^{dR}$, i.e. $\underbrace{\nabla(a \cdot s) = \hbar\, d^{dR} a \otimes s + a \nabla(s)}_{\text{CCR: position and momenta}}$

(3) a $G$-eqv. graded module map $\underline{\psi} : \mathfrak{g}_{[-1]} \otimes \mathcal{E}_\# \longrightarrow \mathcal{E}_\#$ (action of antighosts)

satisfying:

(i) $\underbrace{\nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar\, \nabla_{[v,v']} \;, \quad \nabla_v \underline{\psi}_t = \underline{\psi}_t \nabla_v \;, \quad \underline{\psi}_t \underline{\psi}_{t'} = -\underline{\psi}_{t'} \underline{\psi}_t}_{\text{CCR: momenta and antighosts}}$

(ii) $\underbrace{\partial \underline{\psi}_t + \underline{\psi}_t \partial = \nabla_{\rho^*(t)} + \hbar\, \rho(t)}_{\text{CCR: ghosts and antighosts}}$

(+ technical conditions)

**Morphisms:**

$$\underline{\hom}\Big((\mathcal{E}_\bullet, \nabla, \underline{\psi}), (\mathcal{E}'_\bullet, \nabla', \underline{\psi}')\Big) := \underline{\hom}^{G, \nabla, \underline{\psi}}_{\mathcal{O}(X)[[\hbar]]}(\mathcal{E}_\bullet, \mathcal{E}'_\bullet)$$