The Stack of Yang-Mills Fields

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Outline

1. Generalized smooth spaces via sheaf categories
2. Homotopy theory of presheaves of groupoids and stacks
3. Stack of gauge fields on a manifold
4. Yang-Mills equation and stacky Cauchy problem

Generalized smooth spaces via sheaf categories

Functor of points

- Category of finite-dimensional manifolds Man
- "Test" $M \in$ Man via smooth maps $V \to M$, e.g.
  - $V = \{*\}$ gives points $\{*\} \to M$
  - $V = \mathbb{R}$ gives smooth curves $\mathbb{R} \to M$
- **Technically:** Assign to $M \in$ Man the presheaf (functor of points)

$$\underline{M} := C^\infty(-, M) \ : \ \mathsf{Man}^{\mathrm{op}} \longrightarrow \mathsf{Set} \quad .$$

$\underline{M}(V) = C^\infty(V, M)$ is called the set of V-points.

Crucial observation

By Yoneda lemma, $\underline{(-)} : \mathsf{Man} \to \mathsf{PSh}(\mathsf{Man})$ is fully faithful, i.e.

$$C^\infty(M, M') \ \cong \ \mathrm{Hom}_{\mathrm{PSh}(\mathsf{Man})}\big(\underline{M}, \underline{M'}\big) \quad .$$

Hence, manifolds and their smooth maps can be described equivalently from the functor of points perspective!

Sheaves are better than presheaves!

◇ **Problem:** Given open cover $\{U_i \subseteq M\}$

✓ $$M \xleftarrow{\cong} \mathrm{colim}_{\mathsf{Man}}\Big(\coprod_i U_i \leftleftarrows \coprod_{ij} U_{ij} \Lleftarrow \coprod_{ijk} U_{ijk} \;\; \cdots \Big)$$

ϟ $$\underline{M} \xleftarrow{\ncong} \mathrm{colim}_{\mathrm{PSh}(\mathsf{Man})}\Big(\coprod_i \underline{U_i} \leftleftarrows \coprod_{ij} \underline{U_{ij}} \Lleftarrow \coprod_{ijk} \underline{U_{ijk}} \;\; \cdots \Big)$$

! Solved by restricting to sheaf category $\mathrm{Sh}(\mathsf{Man}) \subseteq \mathrm{PSh}(\mathsf{Man})$.

Def: $X : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf if $\forall$ open covers $\{U_i \subseteq M\}$

$$X(M) \xrightarrow{\cong} \lim_{\mathsf{Set}}\Big(\prod_i X(U_i) \rightrightarrows \prod_{ij} X(U_{ij}) \Rrightarrow \prod_{ijk} X(U_{ijk}) \;\; \cdots \Big)$$

Generalized smooth spaces

We have a fully faithful embedding $\underline{(-)} : \mathsf{Man} \to \mathrm{Sh}(\mathsf{Man})$, i.e. we can equivalently describe manifolds and smooths maps within $\mathrm{Sh}(\mathsf{Man})$.

There are many sheaves $X \in \mathrm{Sh}(\mathsf{Man})$ that **do not** come from manifolds, i.e. $X \ncong \underline{M}$ for any M. These may be called generalized smooth spaces.

Constructions with generalized smooth spaces

- All limits and colimits exist in $\mathrm{Sh}(\text{Man})$. For example, fiber products $X \times_Z Y$ and quotient spaces X/G.
- All exponential objects (mapping spaces) exist in $\mathrm{Sh}(\text{Man})$. For example, field space $\mathrm{Map}(\underline{M}, \underline{N})$ of σ-model: $\mathrm{Map}(\underline{M}, \underline{N})(V) \cong C^\infty(V \times M, N)$
- Tangent bundles $T^{\mathrm{sh}}X \to X$ on all $X \in \mathrm{Sh}(\text{Man})$.

  Sketch: Tangent functor $T : \text{Man} \to \text{Man}$ defines $T^* : \mathrm{Sh}(\text{Man}) \to \mathrm{Sh}(\text{Man})$ via $T^*X(V) = X(TV)$. T^{sh} is left adjoint of T^*. By construction, $T^{\mathrm{sh}}\underline{M} \cong \underline{TM}$.
- Differential forms on all $X \in \mathrm{Sh}(\text{Man})$. Explicitly:
  - Classifying space $\Omega^p \in \mathrm{Sh}(\text{Man})$ given by $\Omega^p : M \mapsto \Omega^p(M)$.
  - Yoneda implies $\omega \in \Omega^p(M) \;\Leftrightarrow\; \omega : \underline{M} \to \Omega^p$ in $\mathrm{Sh}(\text{Man})$.
  - Define p-form ω on X $:\Leftrightarrow$ $\omega : X \to \Omega^p$ in $\mathrm{Sh}(\text{Man})$.

Rem:

1.) Instead of $\mathrm{Sh}(\text{Man})$ we can equivalently take $\mathrm{Sh}(\text{Cart})$ over the full subcategory Cart $\subseteq$ Man given by all $U \cong \mathbb{R}^m$, for some $m \geq 0$.
(The relevant covers in Cart are good open covers.)

2.) These techniques are very useful to study non-linear classical field theories and their Poisson algebras. [Benini,AS:1602.00708] [Khavkine,Schreiber:1701.06238]

Homotopy theory of presheaves of groupoids and stacks

Groupoids

◇ "Spaces" of gauge fields don't have sets but groupoids of points:

$$G\mathbf{Con}(M)(\{*\}) = \begin{cases} \text{Obj: principal } G\text{-bundles over } M \text{ with connection } (A,P) \\ \text{Mor: gauge transformations } h:(A,P)\to(A',P') \end{cases}$$

◇ **New feature:** Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are "the same" not only when isomorphic, but also when equivalent (as categories)!

Ex: $X \times G \to X$ free G-action on set X, then

$$[\,\cdot\,] \,:\, X//G = \begin{cases} \text{Obj: } x\in X \\ \text{Mor: } x \xrightarrow{g} x\,g \end{cases} \longrightarrow \quad X/G = \begin{cases} \text{Obj: } [x]\in X/G \\ \text{Mor: } [x] \xrightarrow{\mathrm{id}} [x] \end{cases}$$

is equivalence but not isomorphism.

◇ Equip category Grpd with a model structure: A morphism $F:\mathcal{G}\to\mathcal{H}$ is a

- weak equivalence if fully faithful and essentially surjective;
- fibration if $\forall x\in\mathcal{G}$ and $h:F(x)\to y$ in $\mathcal{H}$, $\exists\ g:x\to x'$ in $\mathcal{G}$ s.t. $F(g)=h$;
- cofibration if injective on objects.

Presheaves of groupoids and stacks

- "Smooth" groupoids $:=$ Presheaves of groupoids $\mathsf{H} := \mathrm{PSh}(\mathsf{Cart}, \mathsf{Grpd})$.
- For $X : \mathsf{Cart}^{\mathrm{op}} \to \mathsf{Grpd}$, we call $X(V)$ the groupoid of V-points.
- [Hollander:math/0110247] constructed a model structure on H in which
  - $f : X \to Y$ is weak equivalence iff iso of sheaves of homotopy groups;
  - $X \in \mathsf{H}$ is fibrant object iff for all open covers $\{U_i \subseteq U\}$

$$X(U) \xrightarrow{\text{w.e.}} \mathrm{holim}_{\mathsf{Grpd}}\Big(\prod_i X(U_i) \rightrightarrows \prod_{ij} X(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{ijk} X(U_{ijk}) \;\; \cdots \Big)$$

That is a homotopical generalization of the sheaf condition!

Def: A stack is a fibrant object $X \in \mathsf{H}$.

NB: [Hollander:math/0110247] proved that this description of stacks is equivalent to the ones as fibered categories or lax presheaves of groupoids.

Examples of stacks (relevant for gauge theory)

- Every manifold M defines a stack $\underline{M} := C^\infty(-,M) : \mathsf{Cart}^{\mathrm{op}} \to \mathsf{Set} \hookrightarrow \mathsf{Grpd}$.
- Let G be Lie group. Classifying stack of principal G-bundles:

$$\mathrm{B}G(V) \;=\; \begin{cases} \text{Obj:} \;\; * \\ \text{Mor:} \;\; C^\infty(V,G) \ni g : * \longrightarrow * \end{cases}$$

- Classifying stack of principal G-bundles with connections:

$$\mathrm{B}G_{\mathrm{con}}(V) \;=\; \begin{cases} \text{Obj:} \;\; A \in \Omega^1(V,\mathfrak{g}) \\ \text{Mor:} \;\; C^\infty(V,G) \ni g : A \longrightarrow A \triangleleft g = g^{-1}Ag + g^{-1}\mathrm{d}g \end{cases}$$

- Classifying stack of $\mathrm{ad}(G)$-valued differential forms:

$$\Omega^p_{\mathfrak{g}}(V) \;=\; \begin{cases} \text{Obj:} \;\; \omega \in \Omega^p(V,\mathfrak{g}) \\ \text{Mor:} \;\; C^\infty(V,G) \ni g : \omega \longrightarrow \mathrm{ad}_g(\omega) = g^{-1}\omega g \end{cases}$$

NB: Curvature stack map $F : \mathrm{B}G_{\mathrm{con}} \to \Omega^2_{\mathfrak{g}}$:

$$\begin{cases} A \;\longmapsto\; F(A) = \mathrm{d}A + \tfrac{1}{2}[A,A] \\ \big(g : A \to A \triangleleft g\big) \;\longmapsto\; \big(g : F(A) \to \mathrm{ad}_g(F(A))\big) \end{cases}$$

Homotopical constructions with stacks

- ↯ "Ordinary" constructions **do not** preserve weak equivalences, e.g. in topology:

$$\mathrm{colim}_{\mathsf{Top}}\big(\mathbb{D}^n \leftarrow \mathbb{S}^{n-1} \rightarrow \mathbb{D}^n\big) \cong \mathbb{S}^n \not\cong \{*\} \cong \mathrm{colim}_{\mathsf{Top}}\big(\{*\} \leftarrow \mathbb{S}^{n-1} \rightarrow \{*\}\big)$$

- ◇ Model category theory provides tools to construct derived functors.

- ◇ Homotopy fiber product of stacks $X \times^h_Z Y := \mathrm{holim}_{\mathsf{H}}(X \stackrel{f}{\rightarrow} Z \stackrel{g}{\leftarrow} Y)$

$$\big(X \times^h_Z Y\big)(V) = \begin{cases} \text{Obj: } f(x) \stackrel{k}{\longrightarrow} g(y) \text{ in } Z(V) \\ \text{Mor: } \begin{array}{ccc} f(x) & \stackrel{f(h)}{\longrightarrow} & f(x') \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ g(y) & \underset{g(l)}{\longrightarrow} & g(y') \end{array} \text{ in } Z(V) \end{cases}$$

- ◇ Derived mapping stacks $\mathrm{Map}^h(X,Y) := \mathrm{Map}(Q(X),Y)$ for $Y \in \mathsf{H}$ stack. Q is cofibrant replacement and Map is exponential object in H

$$\mathrm{Map}(Z,Y)(V) = \begin{cases} \text{Obj: } F : \underline{V} \times Z \longrightarrow Y \text{ in } \mathsf{H} \\ \text{Mor: } H : \underline{V} \times Z \times \Delta^1 \longrightarrow Y \text{ in } \mathsf{H} \end{cases}$$

Stack of gauge fields on a manifold

Mapping stack: Ordinary vs. derived

◇ **Wanted:** Stack of principal G-bundles with connections on manifold M.

ϟ Ordinary mapping stack $\mathrm{Map}(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ has $\{*\}$-points

$$\mathrm{Map}(\underline{M}, \mathrm{B}G_{\mathrm{con}})(\{*\}) = \begin{cases} \text{Obj: } & A \in \Omega^1(M, \mathfrak{g}) \\ \text{Mor: } & C^\infty(M,G) \ni g : A \longrightarrow A \triangleleft g \end{cases}$$

Only trivial principal G-bundles! The problem is that $\underline{M}$ is not cofibrant!

Lem: Let $\{U_i \subseteq M\}$ be any open cover with all $U_i \cong \mathbb{R}^m$. Then

$$\underline{M} \longleftarrow Q(\underline{M}) := \pi^{\mathrm{oid}}\Big(\coprod_i \underline{U_i} \Longleftarrow \coprod_{ij} \underline{U_{ij}} \Lleftarrow \coprod_{ijk} \underline{U_{ijk}} \quad \cdots \Big)$$

is a cofibrant replacement of $\underline{M}$ in H.

◇ Derived mapping stack $\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ has $\{*\}$-points

$$\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})(\{*\}) = \begin{cases} \text{Obj: } & \big(\{A_i \in \Omega^1(U_i,\mathfrak{g})\}, \{g_{ij} \in C^\infty(U_{ij},G)\}\big) \\ & \text{s.t. } A_i \triangleleft g_{ij} = A_j \;\;\&\;\; g_{ij}\, g_{jk} = g_{ik} \\ \text{Mor: } & \{h_i \in C^\infty(U_i,G)\} : \big(\{A_i\},\{g_{ij}\}\big) \longrightarrow \big(\{A_i'\},\{g_{ij}'\}\big) \\ & \text{s.t. } A_i \triangleleft h_i = A_i' \;\;\&\;\; g_{ij} h_j = h_i g_{ij}' \end{cases}$$

Differential concretification: Motivation

- The groupoid of V-points of $\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ is

$$\begin{cases} \text{Obj:} & \big(\{A_i \in \Omega^1(V \times U_i, \mathfrak{g})\}, \{g_{ij} \in C^\infty(V \times U_{ij}, G)\}\big) \ + \text{conditions} \\ \text{Mor:} & \{h_i \in C^\infty(V \times U_i, G)\} : \big(\{A_i\}, \{g_{ij}\}\big) \longrightarrow \big(\{A_i'\}, \{g_{ij}'\}\big) \ + \text{conditions} \end{cases}$$

which is equivalent to the groupoid of bundles with connections on $V \times M$.

- ϟ $\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ doesn't carry the desired smooth structure where V-points are smoothly V-parametrized bundles with connections on M.

- We have to "kill" the bundles and connections on the test spaces V, which isn't that easy to do in a homotopically well-defined way!

- **Basic ingredient:** Quillen adjunction $\flat : \mathsf{H} \rightleftarrows \mathsf{H} : \sharp$ such that
  - $\flat X(V) = X(\{*\})$ "discretizes" stacks;
  - $\sharp X(V) \cong \mathsf{Grpd}_{\mathsf{H}}\big(\underline{V}, \sharp X\big) \cong \mathsf{Grpd}_{\mathsf{H}}\big(\flat \underline{V}, X\big)$ "evaluates" stacks on discretized test spaces. [cf. Schreiber, cohesive higher topoi]

NB: $\sharp\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ has as V-points discretely V-parametrized families of bundles with connections on M, without any smoothness requirement.

Differential concretification: Construction

- The following concretification construction corrects (for the case of 1-stacks) a previous *erroneous* attempt by [Fiorenza,Rogers,Schreiber:1304.0236].
- **Basic idea:** Start with stack of discretely parametrized families $\sharp\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$ and recover in a 2-step procedure
  1.) smoothly parametrized families of gauge transformations, and
  2.) smoothly parametrized families of bundles with connections.

1.) Homotopy fiber product $P^h \in \mathsf{H}$ of

$$\sharp\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}}) \xrightarrow{\ \sharp\mathrm{forget}\ } \sharp\mathrm{Map}^h(\underline{M}, \mathrm{B}G) \xleftarrow{\ \mathrm{canonical}\ } \mathrm{Map}^h(\underline{M}, \mathrm{B}G)$$

2.) 1-image factorization (fibrant replacement in truncation of H/P^h)

$$G\mathbf{Con}(M) \;:=\; \mathrm{Im}_1\Big(\, \mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}}) \longrightarrow P^h \,\Big)$$

Prop: The groupoid of V-points of $G\mathbf{Con}(M)$ describes smoothly V-parametrized bundles with connections on M. Explicitly:

$$\begin{cases} \text{Obj:} & \big(\{A_i \in \Omega^{0,1}(V \times U_i, \mathfrak{g})\}, \{g_{ij} \in C^\infty(V \times U_{ij}, G)\}\big) \ + \text{conditions (vertical)} \\ \text{Mor:} & \{h_i \in C^\infty(V \times U_i, G)\} : \big(\{A_i\}, \{g_{ij}\}\big) \longrightarrow \big(\{A_i'\}, \{g_{ij}'\}\big) \ + \text{conditions (vertical)} \end{cases}$$

Yang-Mills equation and stacky Cauchy problem

Yang-Mills stack

- Let M be Lorentzian manifold. Relevant stacks for Yang-Mills theory:
  - $G\mathbf{Con}(M)$ is concretification of $\mathrm{Map}^h(\underline{M}, \mathrm{B}G_{\mathrm{con}})$. Smoothly parametrized bundles with connections $(\mathbf{A}, \mathbf{P}) = (\{A_i\}, \{g_{ij}\})$ on M.
  - $\mathbf{\Omega}^p_{\mathfrak{g}}(M)$ is concretification of $\mathrm{Map}^h(\underline{M}, \Omega^p_{\mathfrak{g}})$. Smoothly parametrized bundles with p-form valued sections of adjoint bundle $(\boldsymbol{\omega}, \mathbf{P})$ on M.
  - $G\mathbf{Bun}(M) := \mathrm{Map}^h(\underline{M}, \mathrm{B}G)$. Smoothly parametrized bundles $\mathbf{P}$ on M.
- Relevant stack morphisms:
  - $\mathbf{0}_M : G\mathbf{Bun}(M) \to \mathbf{\Omega}^p_{\mathfrak{g}}(M)\,,\ \mathbf{P} \mapsto (\mathbf{0}, \mathbf{P})$ assigns zero-sections.
  - $\mathbf{YM}_M : G\mathbf{Con}(M) \to \mathbf{\Omega}^1_{\mathfrak{g}}(M)\,,\ (\mathbf{A}, \mathbf{P}) \mapsto \big(\{\delta^{\mathrm{vert}}_{A_i} F^{\mathrm{vert}}(A_i)\}, \{g_{ij}\}\big)$ is Yang-Mills operator.

Def: The Yang-Mills stack $G\mathbf{Sol}(M)$ is the homotopy fiber product of

$$G\mathbf{Con}(M) \xrightarrow{\ \mathbf{YM}_M\ } \mathbf{\Omega}^1_{\mathfrak{g}}(M) \xleftarrow{\ \mathbf{0}_M\ } G\mathbf{Bun}(M)$$

Prop: The groupoid of V-points describes smoothly V-parametrized solutions of the Yang-Mills equation. Explicitly: $(\mathbf{A}, \mathbf{P})$ s.t. $\delta^{\mathrm{vert}}_{A_i} F^{\mathrm{vert}}(A_i) = 0$.

Stacky Cauchy problem

◇ Given Cauchy surface $\Sigma \subseteq M$, there exists map of stacks $\mathrm{data}_\Sigma : G\mathbf{Sol}(M) \to G\mathbf{Data}(\Sigma)$ which assigns initial data.

Def: The stacky Cauchy problem is well-posed if data_Σ is a weak equivalence.

Theorem [Benini,AS,Schreiber]

The stacky Yang-Mills Cauchy problem is well-posed if and only if the following hold true, for all $V \in$ Cart:

1. For all $(\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(V)$, there exists $(\mathbf{A}, \mathbf{P})$ in $G\mathbf{Sol}(M)(V)$ and iso $\mathbf{h}^\Sigma : \mathrm{data}_\Sigma(\mathbf{A}, \mathbf{P}) \to (\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(V)$.
2. For any other iso $\mathbf{h}'^\Sigma : \mathrm{data}_\Sigma(\mathbf{A}', \mathbf{P}') \to (\mathbf{A}^\Sigma, \mathbf{E}, \mathbf{P}^\Sigma)$ in $G\mathbf{Data}(\Sigma)(V)$, there exists unique iso $\mathbf{h} : (\mathbf{A}, \mathbf{P}) \to (\mathbf{A}', \mathbf{P}')$ in $G\mathbf{Sol}(M)(V)$, such that $\mathbf{h}'^\Sigma \circ \mathrm{data}_\Sigma(\mathbf{h}) = \mathbf{h}^\Sigma$.

! Interesting smoothly V-parametrized Cauchy problems! To the best of my knowledge, results only known for $V = \{*\}$ [Chrusciel,Shatah; Choquet-Bruhat].