Mapping spaces and automorphism groups of toric noncommutative spaces

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Bayrischzell Workshop 2016
Quantum spacetime structures: Dualities and new geometries
Bayrischzell, April 29 – May 3, 2016.

Based on ongoing joint work with Gwendolyn E. Barnes and Richard J. Szabo.

Background and motivation

◇ *Observation:*

- Deformed NC algebras A_Θ typically have fewer automorphisms than their underlying commutative algebras A

⇝ "NC symmetry breaking" / "quantum rigidity"

◇ *Example:*

- Commutative plane: $\mathrm{Aut}(C^\infty(\mathbb{R}^{2n})) \simeq \{$diffeomorphisms of $\mathbb{R}^{2n}\}$
- Moyal plane: $\mathrm{Aut}(C^\infty(\mathbb{R}^{2n})_\Theta) \simeq \{$symplectomorphisms of $(\mathbb{R}^{2n}, \omega = \Theta^{-1})\}$

◇ *Problem in NC gauge theory:*

Let A_Θ be total space of a NC principal bundle. Then the gauge group $\mathrm{Gau}(A_\Theta) \subseteq \mathrm{Aut}(A_\Theta)$ is typically too small [see e.g. Beggs, Majid].

◇ *Goals of my talk:*

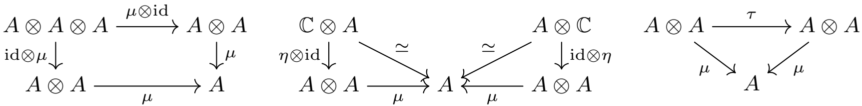
1.) Construct 'internalized' automorphism groups for toric NC spaces via sheaf theoretical techniques

2.) Compare the Lie algebras of these groups to 'internalized' infinitesimal automorphisms (braided derivations)

3.) Application to the gauge group in NC gauge theory

H-comodules and comodule algebras

- $H = \mathcal{O}(\mathbb{T}^n)$ is torus Hopf algebra; vector space basis $t_{\mathbf{m}}$ “ $= e^{i\,\mathbf{m}\,\phi}$ ”, $\mathbf{m} \in \mathbb{Z}^n$
- Cotriangular structure $R(t_{\mathbf{m}} \otimes t_{\mathbf{n}}) = e^{i\,\mathbf{m}\,\Theta\,\mathbf{n}}$ with $n \times n$-matrix Θ
- ${}^H\mathscr{M}$ is category of left H-comodules $\rho^V : V \to H \otimes V\,,~ v \mapsto v_{(-1)} \otimes v_{(0)}$
- ${}^H\mathscr{M}$ is symmetric tensor category, i.e. we have commutativity constraints $\tau : V \otimes W \to W \otimes V\,,~ v \otimes w \mapsto R(w_{(-1)} \otimes v_{(-1)})\, w_{(0)} \otimes v_{(0)}$
- ${}^H\mathscr{A}$ is category of (finitely presented) algebra objects (A, μ, η) in ${}^H\mathscr{M}$



Ex: 1.) $A_{\mathbb{S}^{2N-1}_\Theta} = \mathrm{Free}(x_1,\dots,x_N,x_1^*,\dots,x_N^*)/(\sum_a x_a^*\, x_a - \mathbb{1})$ with
$x_a\, x_b = e^{i\,\mathbf{m}_b\,\Theta\,\mathbf{m}_a}\, x_b\, x_a$ and $x_a\, x_b^* = e^{-i\,\mathbf{m}_b\,\Theta\,\mathbf{m}_a}\, x_b^*\, x_a$

2.) $A_{\mathbb{S}^{2N}_\Theta} = \mathrm{Free}(x_1,\dots,x_N,x_1^*,\dots,x_N^*,y)/(\sum_a x_a^*\, x_a + y^2 - \mathbb{1})$ with
commutation relations as above and $y\, x_a^{(*)} = x_a^{(*)}\, y$

Toric noncommutative spaces

Def: Category of toric NC spaces ${}^{H}\mathscr{S} := ({}^{H}\mathscr{A})^{\mathrm{op}}$.

- Objects X_A are specified by objects A in ${}^{H}\mathscr{A}$
- Morphisms $f : X_A \to X_B$ are specified by ${}^{H}\mathscr{A}$-morphisms $f^* : B \to A$

! Going to the opposite category is useful for geometric intuition!

Lem: ${}^{H}\mathscr{S}$ has products $X_A \times X_B = X_{A \sqcup B}$ and pullbacks $X_A \times_{X_C} X_B = X_{A \sqcup_C B}$

$$\begin{array}{ccc} X_A \times_{X_C} X_B & \dashrightarrow & X_B \\ \downarrow & & \downarrow g \\ X_A & \xrightarrow[f]{} & X_C \end{array} \qquad \text{corresponds to} \qquad \begin{array}{ccc} C & \xrightarrow{g^*} & B \\ f^* \downarrow & & \downarrow \\ A & \dashrightarrow & A \sqcup_C B \end{array}$$

NB: The category ${}^{H}\mathscr{S}$ is too small for our purposes. It does not contain objects ${X_B}^{X_A}$ that describe the 'space of mappings' from X_A to X_B

[Technically $- \times X_A : {}^{H}\mathscr{S} \to {}^{H}\mathscr{S}$ does not have a right adjoint functor.]

◇ *Way out:* Enlarge the category ${}^{H}\mathscr{S}$ in a controlled way!

$\rightsquigarrow$ generalized toric NC spaces $\widehat{=}$ sheaves on ${}^{H}\mathscr{S}$

${}^{H}\mathscr{S}$-Zariski covering families

Def: An ${}^{H}\mathscr{S}$-Zariski covering family is finite family of ${}^{H}\mathscr{S}$-morphisms

$$\big\{ f_i : X_{A_i} \longrightarrow X_A \big\}$$

such that

(i) $A_i = A[s_i^{-1}]$ is localization of A w.r.t. H-coinvariant $s_i \in A$;

(ii) $f_i^* : A \to A[s_i^{-1}]$ is the canonical ${}^{H}\mathscr{A}$-morphism;

(iii) there exists $a_i \in A$ such that $\sum_i a_i\, s_i = \mathbb{1}$.

Ex: For $A_{\mathbb{S}^{2N}_\Theta} = \mathrm{Free}(x_1, \dots, x_N, x_1^*, \dots, x_N^*, y)/(\sum_a x_a^*\, x_a + y^2 - \mathbb{1})$ choose $s_\pm = \frac{1}{2}(1 \pm y)$ (localizes away from the South/North Pole)

Prop: ${}^{H}\mathscr{S}$-Zariski covering families are stable under pullbacks.

In particular, $X_{A[s_i^{-1}]} \times_{X_A} X_{A[s_j^{-1}]} \simeq X_{A[s_i^{-1}, s_j^{-1}]}$.

- ◇ *Loosely speaking:* The overlap of two of our patches has 'function algebra' given by the localization $A[s_i^{-1}, s_j^{-1}]$ w.r.t. both elements s_i and s_j

Generalized toric NC spaces

Def: Category of generalized toric NC spaces ${}^H\mathscr{G} := \mathrm{Sh}({}^H\mathscr{S})$.

- Objects are functors $Y : {}^H\mathscr{S}^{\mathrm{op}} \to$ Set satisfying sheaf condition

$$Y\big(X_A\big) \longrightarrow \prod_i Y\big(X_{A[s_i^{-1}]}\big) \rightrightarrows \prod_{i,j} Y\big(X_{A[s_i^{-1},\, s_j^{-1}]}\big)$$

- Morphisms are natural transformations $f : Y \to Y'$

◇ Toric NC spaces embed via (fully faithful) Yoneda embedding ${}^H\mathscr{S} \to {}^H\mathscr{G}$

- On objects X_A we have $\underline{X_A} := \mathrm{Hom}_{{}^H\mathscr{S}}(-, X_A)$
- On morphisms $f : X_A \to X_B$ we have $\underline{f} := \mathrm{Hom}_{{}^H\mathscr{S}}(-, f) : \underline{X_A} \to \underline{X_B}$

? *Interpretation:*

- $\underline{X_A}(X_B) = \mathrm{Hom}_{{}^H\mathscr{S}}(X_B, X_A)$ described all possible ways how X_B can be mapped into the toric NC space X_A
- Interpret $Y(X_B)$ as the set of all possible ways how X_B can be mapped into the generalized toric NC space Y (Grothendieck's "functor of points")

! Warning: Generalized toric NC spaces are **not** described by algebras!

Categorical properties of ${}^{H}\mathscr{G}$ / Mapping spaces

- The category of generalized toric NC spaces is very good (technically called Grothendieck topos). In particular:
  - ${}^{H}\mathscr{G}$ has all limits (e.g. fibred products) and colimits (e.g. quotient spaces)
  - ${}^{H}\mathscr{G}$ has exponential objects Z^Y ('spaces of mappings from Y to Z')
- Let's have a closer look at $\underline{X_B}^{\underline{X_A}}$ for two ordinary toric NC spaces
- The functor of points is

$$\underline{X_B}^{\underline{X_A}}(X_C) = \mathrm{Hom}_{{}^{H}\mathscr{S}}(X_C \times X_A, X_B) = \mathrm{Hom}_{{}^{H}\mathscr{A}}(B, C \sqcup A)$$

i.e. $\{$ maps $X_C \longrightarrow \underline{X_B}^{\underline{X_A}}$ $\}$ $\simeq$ $\{$ maps $X_C \times X_A \longrightarrow X_B$ $\}$ (adjunction!)

Rem:
- Global points $\{*\} \longrightarrow \underline{X_B}^{\underline{X_A}}$ are precisely ${}^{H}\mathscr{S}$-morphisms $X_A \to X_B$
- Generalized points $X_C \longrightarrow \underline{X_B}^{\underline{X_A}}$ capture additional information

! That's the key to our original problem to find a bigger automorphism group for toric NC spaces!

Toy example: Endomorphisms of a 2-dim. toric NC plane

◇ Consider $A = \mathrm{Free}(x,y)$ with $x\,y = e^{i\,\Theta}\,y\,x$

◇ ${}^{H}\mathscr{A}$-morphism $\kappa : A \to A$ has to satisfy

$$\kappa(x)\,\kappa(y) = \kappa(x\,y) = e^{i\,\Theta}\;\kappa(y\,x) = e^{i\,\Theta}\;\kappa(y)\,\kappa(x)$$

$\Rightarrow$ $\mathrm{Hom}_{{}^{H}\mathscr{S}}(X_A, X_A) \simeq \big\{\kappa : A \to A \,:\, \kappa(x) \in x\,\mathbb{C}$ and $\kappa(y) \in y\,\mathbb{C}\big\}$

◇ Mapping space is better. For $C = \mathrm{Free}(z)$ with $\rho : z \mapsto t_{(-1,0)} \otimes z$

$$\underline{X_A}^{\underline{X_A}}(X_C) \simeq \Big\{\kappa : A \to C \sqcup A \,:\, \kappa(x) \in \sum_k z^k \otimes x^{k+1}\,\mathbb{C} \text{ and } \kappa(y) \in y\,\mathbb{C}\Big\}$$

and for $C = \mathrm{Free}(z)$ with $\rho : z \mapsto t_{(0,-1)} \otimes z$

$$\underline{X_A}^{\underline{X_A}}(X_C) \simeq \Big\{\kappa : A \to C \sqcup A \,:\, \kappa(x) \in x\,\mathbb{C} \text{ and } \kappa(y) \in \sum_k z^k \otimes y^{k+1}\,\mathbb{C}\Big\}$$

! Varying C allows us to capture generic polynomial mappings $A \to A$

Loosely speaking: Additional mappings are encoded in generalized points!

Automorphism group and its Lie algebra

◇ $\underline{X_A}^{\underline{X_A}}$ is a monoid object in ${}^H\mathscr{G}$

$$e : \{*\} \longrightarrow \underline{X_A}^{\underline{X_A}} \quad , \qquad \bullet : \underline{X_A}^{\underline{X_A}} \times \underline{X_A}^{\underline{X_A}} \longrightarrow \underline{X_A}^{\underline{X_A}}$$

Def: Automorphism group $\mathrm{Aut}(X_A)$ is subobject of invertibles in the monoid object $\underline{X_A}^{\underline{X_A}}$.

Prop: $\mathrm{Aut}(X_A)$ is group object in ${}^H\mathscr{G}$ ("generalized toric NC Lie group")

◇ Tangent bundle: mapping "infinitesimal line" $D = X_{\mathrm{Free}(x)/(x^2)}$ to $\mathrm{Aut}(X_A)$

$$T\mathrm{Aut}(X_A) = \mathrm{Aut}(X_A)^D \xrightarrow{\quad \pi \quad} \mathrm{Aut}(X_A)$$

Prop: The pullback

$$\begin{array}{ccc} T_e\mathrm{Aut}(X_A) & \dashrightarrow & T\mathrm{Aut}(X_A) \\ \downarrow & & \downarrow \pi \\ \{*\} & \xrightarrow[e]{} & \mathrm{Aut}(X_A) \end{array}$$

produces Lie algebra object $T_e\mathrm{Aut}(X_A)$ in ${}^H\mathscr{G}$ (infinitesimal automorphisms).

What did we get?

- The abstract machinery produced an automorphism group and its Lie algebra for toric NC spaces X_A (both are objects in ${}^H\mathscr{G}$, i.e. sheaves).

- $\mathrm{Aut}(X_A)$ has no description in elementary terms (no Hopf algebra!), but $T_e\mathrm{Aut}(X_A)$ can be identified with the braided derivations of A.

- $\mathrm{der}(A)$ are linear maps $L : A \to A$ satisfying braided Leibniz rule

$$L(a\,a') = L(a)\,a' + R(a_{(-1)} \otimes L_{(-1)})\;a_{(0)}\,L_{(0)}(a')$$

- $\mathrm{der}(A)$ is Lie algebra object in ${}^H\mathscr{M}$

- How to compare with Lie algebra object $T_e\mathrm{Aut}(X_A)$ in ${}^H\mathscr{G}$??

Thm: There exists fully faithful embedding $j : {}^H\mathscr{M} \to \mathrm{Mod}_C({}^H\mathscr{G})$ such that $j(\mathrm{der}(A)) \simeq T_e\mathrm{Aut}(X_A)$.

- *Sketch:* j is given by $j(V)(X_A) = (A \otimes V)^{\mathrm{coH}}$. Proving fully faithfulness requires detailed understanding of ${}^H\mathscr{M}$, which we have for $H = \mathcal{O}(\mathbb{T}^n)$.

Application to NC gauge theory

- G group object in ${}^H\mathscr{S}$ (i.e. $G = X_Q$ with Q Hopf algebra) and $r : P \times G \to P$ right G-action in ${}^H\mathscr{S}$ (i.e. $P = X_A$ with Q-comod. alg. A).

- Quotient is coequalizer $P \times G \overset{r}{\underset{\mathrm{pr}_1}{\rightrightarrows}} P \longrightarrow P/G$ in ${}^H\mathscr{S}$ (exists!)

Def: $r : P \times G \to P$ is toric NC principal bundle over P/G if we have iso

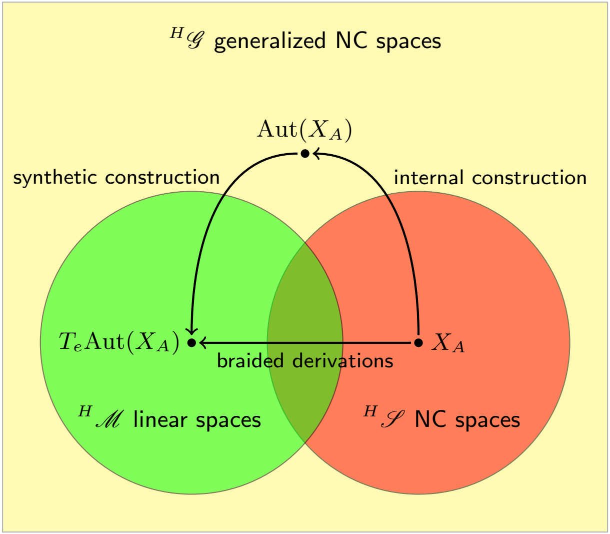
$$\begin{array}{ccccc} & & P \times G & & \\ & \overset{r}{\swarrow} & \downarrow \simeq & \overset{\mathrm{pr}_1}{\searrow} & \\ P & \longleftarrow & P \times_{P/G} P & \longrightarrow & P \end{array}$$

- $\mathrm{Gau}(P)$ is subobject of $\mathrm{Aut}(P)$ preserving G-action and base space P/G

- Lie algebra $T_e\mathrm{Gau}(P)$ has elementary description in terms of $\mathrm{der}(A)$ compatible with Q-coaction and coinvariants $B = A^{\mathrm{co}Q}$ (notice $X_B = P/G$)

$$(L \otimes \mathrm{id}_Q)\delta^Q(a) = \delta^Q(L(a)) \ ,$$
$$L(b) = 0 \ .$$

Summary in one picture



$\mathrm{Aut}(X_A)$ has no elementary description (no Hopf algebras!)