# Category theoretical description of matter and gauge QFTs

Alexander Schenkel

Mathematics Department, University of Wuppertal

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(i) [arXiv:1210.3457 [math-ph]] to appear in Annales Henri Poincaré

(ii) second paper will be soon on the arXiv

# Motivation

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- Locally covariant QFT is a modern formulation of QFT on curved spacetimes. Instead of working on one spacetime, the goal is to coherently describe QFTs on **all** spacetimes. This can be achieved by employing techniques from category theory.
- This approach did not only set up an axiomatic approach to QFT on curved spacetimes, extending the Haag-Kastler approach, but it has already led to deep results and insights, for example:
  - Renormalization on generic spacetimes [Hollands,Wald;Brunetti,Dütsch,Fredenhagen,Rejzner]
  - Spin-statistics theorem on generic spacetimes [Verch]
  - Semiclassical Einstein equations [Dappiaggi,Hack,Pinamonti]
- In my talk I have two goals:
  - I will review the framework of locally covariant QFT, as it has been developed by Brunetti, Fredenhagen and Verch [arXiv:math-ph/0112041].
  - I will consider modifications of this setting required to include gauge theories and construct a functor describing quantized Abelian principal connections.

# Outline

1. Locally covariant QFT: A new paradigm

2. General linear and affine matter QFTs

3. Locally covariant QFT and gauge symmetry

4. Construction of the Maxwell functor

5. Conclusions and outlook

# Locally covariant QFT: A new paradigm

# The essentials of QFT

The following wish-list characterizes in my opinion in a minimalistic way what a QFT should do:

**I.** A QFT should not only be defined on one, but on all spacetimes:

General relativity allows for many different solutions, which all can be of physical significance. Even more, the QFT itself can backreact on spacetime due to its energy-momentum tensor $\rightarrow$ spacetime has to be determined dynamically in presence of the QFT!

**II.** A QFT should associate to every spacetime the local observables that can be measured in it:

Since the choice of quantum state is non-canonical it is important to associate abstract algebras and not algebras in representations!

**III.** Larger spacetimes should have more observables, isometric ones the same:

If there exists an isometric embedding $\chi : N \rightarrow M$ of a spacetime $N$ into a spacetime $M$, then the observables in $N$ should be embeddable in the observables in $M$. In the case $\chi$ is an isometry (isomorphism), then the observables in $N$ and $M$ should be isomorphic.

# Locally covariant QFT: Preliminaries

Brunetti, Fredenhagen and Verch have turned the wish-list into an axiomatic framework for QFT [arXiv:math-ph/0112041]. This framework makes heavy use of categories, functors and natural transformations, since this is the natural language to formalize the wish-list:

- We required spacetimes and isometric embeddings of spacetimes. These two concepts can be unified in a category of spacetimes Man.
- On the other hand we required algebras and embeddings of algebras. Again, we can unify the two concepts in a category of algebras Alg.
- Since a QFT should associate algebras to spacetimes and algebra embeddings to isometric embeddings, it is natural to formulate it as a covariant functor $\mathfrak{A} : \mathsf{Man} \to \mathsf{Alg}$.

But before going into the details, let me repeat some definitions in category theory . . .

# Notions in category theory: Categories

**Def:** A category C consists of the following data:

- A class $\mathrm{Obj}(\mathsf{C})$ whose members are called **objects**
- For any two objects $A, B$ there is a (possibly empty) set $\mathrm{Mor}(A, B)$ whose elements are called **morphisms**
- For any three objects $A, B, C$ there is a map (**composition**)

$$\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C) \ , \quad (g, f) \mapsto g \circ f$$

such that the following axioms are fulfilled:

(1) If the pairs of objects $(A, B)$ and $(A', B')$ are not equal, then the sets $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}(A', B')$ are disjoint.

(2) For every object $A$ there is an element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$, such that for all objects $B$ and all $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$

$$f \circ \mathrm{id}_A = f \ , \quad \mathrm{id}_A \circ g = g$$

(3) The composition law is associative, i.e. for any four objects $A, B, C, D$ and any $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $h \in \mathrm{Mor}(C, D)$

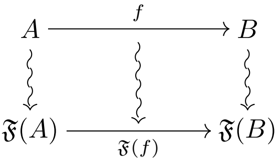
$$(h \circ g) \circ f = h \circ (g \circ f)$$

# Notions in category theory: Functors

**Def:** Let C and D be two categories. A covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{D}$ is a mapping that:

- associates to each object $A$ in C an object $\mathfrak{F}(A)$ in D
- associates to each morphism $f \in \mathrm{Mor}(A, B)$ a morphism $\mathfrak{F}(f) \in \mathrm{Mor}\big(\mathfrak{F}(A), \mathfrak{F}(B)\big)$, such that
  (1) for any object $A$ in C, $\mathfrak{F}(\mathrm{id}_A) = \mathrm{id}_{\mathfrak{F}(A)}$
  (2) for any objects $A, B, C$ in C and any $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$

Picture:



# Notions in category theory: Natural transformations

**Def:** Let C and D be two categories and $\mathfrak{F}, \mathfrak{G} : \mathsf{C} \to \mathsf{D}$ be two covariant functors. A natural transformation $\eta : \mathfrak{F} \Rightarrow \mathfrak{G}$ associates to every object $A$ in C a morphism $\eta_A \in \mathrm{Mor}(\mathfrak{F}(A), \mathfrak{G}(A))$ in D, such that for all objects $A, B$ in C and morphisms $f \in \mathrm{Mor}(A, B)$

$$\begin{array}{ccc} \mathfrak{F}(A) & \xrightarrow{\eta_A} & \mathfrak{G}(A) \\ {\scriptstyle \mathfrak{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathfrak{G}(f)} \\ \mathfrak{F}(B) & \xrightarrow{\eta_B} & \mathfrak{G}(B) \end{array}$$

**NB:** You can interpret natural transformations as morphisms of functors.

# The categories of locally covariant QFT

Man: The objects in $\mathrm{Obj}(\mathsf{Man})$ are oriented and time-oriented globally hyperbolic Lorentzian manifolds $(M, g)$.

A morphism between two objects $(M_1, g_1)$ and $(M_2, g_2)$ is an orientation and time-orientation preserving isometric embedding $f : M_1 \to M_2$, whose image $f[M_1] \subseteq M_2$ is causally convex.

Alg: The objects in $\mathrm{Obj}(\mathsf{Alg})$ are unital $*$-algebras $A$.

A morphism between two objects $A_1$ and $A_2$ is an injective unital $*$-algebra homomorphism $\pi : A_1 \to A_2$.

**Def:** (i) A locally covariant QFT is a covariant functor $\mathfrak{A} : \mathsf{Man} \to \mathsf{Alg}$.

(ii) A locally covariant QFT $\mathfrak{A}$ is causal, if whenever there are morphisms $f_1 : (M_1, g_1) \to (M, g)$ and $f_2 : (M_2, g_2) \to (M, g)$ such that $f_1[M_1]$ and $f_2[M_2]$ are causally disjoint in $M$, then

$$\Big[\mathfrak{A}(f_1)\big(\mathfrak{A}(M_1, g_1)\big), \mathfrak{A}(f_2)\big(\mathfrak{A}(M_2, g_2)\big)\Big] = \{0\} \ .$$

(iii) A locally covariant QFT $\mathfrak{A}$ satisfies the time-slice axiom, if whenever $f : (M_1, g_1) \to (M_2, g_2)$ is such that $f[M_1]$ contains a Cauchy surface of $M_2$, then $\mathfrak{A}(f) : \mathfrak{A}(M_1, g_1) \to \mathfrak{A}(M_2, g_2)$ is an isomorphism.

# Locally covariant quantum fields

- In QFT on a single spacetime $(M,g)$, quantum fields are operator valued distributions $\Phi : C_0^\infty(M) \to \mathfrak{A}(M,g)$, $\Phi(\varphi) \stackrel{\text{formal}}{=} \int_M \Phi(x)\,\varphi(x)\,\mathrm{vol}_M$.
- In locally covariant QFT we would like to have a family $\Phi = \{\Phi_{(M,g)}\}$ of such operator valued distributions, which satisfies certain covariance properties. This is exactly a natural transformation $\Phi : \mathfrak{D} \Rightarrow \mathfrak{A}$ between the functor $\mathfrak{D}$ : Man $\to$ Lin describing test-sections, $\mathfrak{D}(M,g) = C_0^\infty(M)$ and $\mathfrak{D}(f) = f_* : C_0^\infty(M_1) \to C_0^\infty(M_2)$, and the functor $\mathfrak{A}$ : Man $\to$ Alg $\subset$ Lin.

**Def:** A locally covariant quantum field is a natural transformation $\Phi : \mathfrak{D} \Rightarrow \mathfrak{A}$, i.e. a family of linear maps $\Phi_{(M,g)} : \mathfrak{D}(M,g) \to \mathfrak{A}(M,g)$, such that

$$\begin{array}{ccc} \mathfrak{D}(M_1,g_1) & \xrightarrow{\Phi_{(M_1,g_1)}} & \mathfrak{A}(M_1,g_1) \\ \downarrow{\scriptstyle \mathfrak{D}(f)} & & \downarrow{\scriptstyle \mathfrak{A}(f)} \\ \mathfrak{D}(M_2,g_2) & \xrightarrow{\Phi_{(M_2,g_2)}} & \mathfrak{A}(M_2,g_2) \end{array}$$

In other words: $\mathfrak{A}(f)\big(\Phi_{(M_1,g_1)}(\varphi)\big) = \Phi_{(M_2,g_2)}(f_*(\varphi))$

# Example: The Klein-Gordon field I

It is time to provide an example! Let us start with the simplest model, the real Klein-Gordon field.

- On any object $(M, g)$ in Man we can consider the Klein-Gordon operator $P_{(M,g)} = \Box_{(M,g)} - m^2 - \xi\, R$.

- There exist unique retarded/advanced Green's operators $\Delta^{\pm}_{(M,g)}$ and the classical solution theory can be summarized in the following exact complex

$$0 \longrightarrow C^\infty_0(M) \xrightarrow{P_{(M,g)}} C^\infty_0(M) \xrightarrow{\Delta_{(M,g)}} C^\infty_{\mathrm{sc}}(M) \xrightarrow{P_{(M,g)}} C^\infty_{\mathrm{sc}}(M)$$

- We associate the classical phasespace $V_{(M,g)} := C^\infty_0(M)/P_{(M,g)}[C^\infty_0(M)]$ and equip it with the symplectic structure

$$\omega_{(M,g)} : V_{(M,g)} \times V_{(M,g)} \to \mathbb{R}\ , \quad ([\varphi],[\psi]) \mapsto \omega_{(M,g)}([\varphi],[\psi]) = \int_M \varphi\, \Delta_{(M,g)}(\psi)\, \mathrm{vol}_M$$

- One shows that $\mathfrak{PhSp}(M,g) = (V_{(M,g)}, \omega_{(M,g)})$ and $\mathfrak{PhSp}(f) = f_\ast : (V_{(M_1,g_1)}, \omega_{(M_1,g_1)}) \to (V_{(M_2,g_2)}, \omega_{(M_2,g_2)})$ defines a covariant functor $\mathfrak{PhSp}$ : Man $\to$ Sympl.

$\Rightarrow$ The classical Klein-Gordon field is functorial.

# Example: The Klein-Gordon field II

- There is the canonical commutation relation functor $\mathfrak{CCR} : \mathsf{Sympl} \to \mathsf{Alg}$. To any symplectic vector space $(V,\omega)$ it associates the unital $*$-algebra $\mathfrak{CCR}(V,\omega)$ generated by hermitian symbols $\Phi_{(V,\omega)}(v)$, $v \in V$, modulo the ideal generated by the relations
  - $\Phi_{(V,\omega)}(\alpha\, v + \beta\, w) = \alpha\, \Phi_{(V,\omega)}(v) + \beta\, \Phi_{(V,\omega)}(w)$
  - $[\Phi_{(V,\omega)}(v), \Phi_{(V,\omega)}(w)] = i\hbar\, \omega(v,w)\, 1$

  To any symplectic map $L : (V_1,\omega_1) \to (V_2,\omega_2)$ it associates the $*$-algebra homomorphism $\mathfrak{CCR}(L)$ specified by $\mathfrak{CCR}(L)(\Phi_{(V_1,\omega_1)}(v)) := \Phi_{(V_2,\omega_2)}(Lv)$.

- Defining $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp} : \mathsf{Man} \to \mathsf{Alg}$ we obtain a locally covariant QFT, that is causal and satisfies the time-slice axiom.

- There is a locally covariant quantum field $\Phi : \mathfrak{D} \Rightarrow \mathfrak{A}$ specified by, for all objects $(M,g)$,
  $\Phi_{(M,g)} : C_0^\infty(M) \to \mathfrak{A}(M,g)\,,~ \varphi \mapsto \Phi_{(M,g)}(\varphi) := \Phi_{\mathfrak{PhSp}(M,g)}(\varphi)$.

**Rem:** A proper extension of this algebra by Wick polynomials allows for a variety of other locally covariant quantum fields, e.g. the Wick polynomials or the stress-energy tensor.

# General linear and affine matter QFTs

# Problems with general matter QFTs

- Can we also include general linear matter fields into the framework of locally covariant QFT?
- Let us for example consider a Dirac field described by the massive Dirac operator $\not{D} + m = \gamma^\mu \nabla_\mu + m$.
- Problem: Neither the Dirac operator, nor the spinor bundle on which it acts, can be canonically associated to a Loretzian manifold. We require a spin bundle on $(M, g)$!
- Strategy: Increase the structure in the geometric category Man to describe more general QFTs.

**Ex:** The Dirac field is a locally covariant QFT in the sense that it is a covariant functor from the category of spin manifolds SpMan to Alg [Sanders;Dappiaggi,Hack,Pinamonti].

# A unified description of all linear matter QFTs

- ◇ Bär, Ginoux and Pfäffle have constructed all linear bosonic and fermionic matter QFTs with respectively one functor [arXiv:0806.1036 [math.DG], arXiv:1104.1158 [math-ph]].
- ◇ Their trick was to consider the following categories $\mathsf{GlobHypGreen}^{\mathrm{bos/ferm}}$:
  - An object is a triple $(M, V, P)$, where $M$ is an oriented and time-oriented globally hyperbolic Lorentzian manifold, $V$ is a real vector bundle over $M$ with inner product (symplectic structure) $\langle\ ,\ \rangle$ and $P$ is a formally self-adjoint Green-hyperbolic operator acting on sections of $V$.
  - A morphism between two objects $(M_1, V_1, P_1)$ and $(M_2, V_2, P_2)$ is a vector bundle map $(f : V_1 \to V_2, \underline{f} : M_1 \to M_2)$, such that $f$ is an isomorphism on the fibres that preserves $\langle\ ,\ \rangle$, $\underline{f}$ is as in Man and the diagram commutes

$$\begin{array}{ccc} \Gamma^\infty(V_2) & \xrightarrow{\ P_2\ } & \Gamma^\infty(V_2) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \Gamma^\infty(V_1) & \xrightarrow{\ P_1\ } & \Gamma^\infty(V_1) \end{array}$$

**Thm:** There exist covariant functors $\mathfrak{A}_{\mathrm{lin}}^{\mathrm{bos/ferm}} : \mathsf{GlobHypGreen}^{\mathrm{bos/ferm}} \to \mathsf{Alg}$ satisfying the causality property and the time-slice axiom.

# A unified description of all affine matter QFTs

- ◇ Vector bundles and linear operators are not always appropriate for describing field theories. For example, gauge fields have to be described by affine bundles (see later) and inhomogeneous matter fields $P(s) = J$ have an affine space structure on the level of the dynamics.

- ◇ In [arXiv:1210.3457 [math-ph]] we have proposed the categories $\mathsf{GlobHypAffGreen}^{\mathrm{bos/ferm}}$ where objects are triples $(M, A, P)$, with $M$ being a spacetime, $A$ an affine bundle and $P$ an affine Green-hyperbolic operator. Morphisms are compatible affine bundle maps.

**Thm:** There exist covariant functors $\mathfrak{A}_{\mathrm{aff}}^{\mathrm{bos/ferm}} : \mathsf{GlobHypAffGreen}^{\mathrm{bos/ferm}} \to \mathsf{Alg}$ satisfying the causality property and the time-slice axiom.

**Rem:** The functor $\mathfrak{A}_{\mathrm{lin}}^{\mathrm{bos/ferm}}$ is a mapping to the full subcategory of unital $*$-algebras with a trivial center, while the functor $\mathfrak{A}_{\mathrm{aff}}^{\mathrm{bos/ferm}}$ is not!

- ◇ Physically, this means that affine QFTs contain also classical degrees of freedom, which in the case of inhomogeneous matter fields can be interpreted as the source term $J$. For gauge theories we will identify topological charges as elements in the center.

# Locally covariant QFT and gauge symmetry

# Generalities I

- The correct geometric framework for describing gauge theories is that of principal bundles $G \hookrightarrow P \xrightarrow{\pi} M$.
- On any principal bundle we have (by definition) an action of the structure group $G$ on the total space $P$ from the right, $r : P \times G \to P\,,\ (p, g) \mapsto p\, g$.
- A morphism of between principal bundles is a smooth map $f : P_1 \to P_2$ that is $G$-equivariant, $f(p\, g) = f(p)\, g$. This determines a unique smooth map $\underline{f} : M_1 \to M_2$ between the base spaces, such that $\pi_2 \circ f = \underline{f} \circ \pi_1$.
- A gauge transformation is an automorphism $f : P \to P$, such that $\underline{f} = \mathrm{id}_M$.
- Given any representation $(\rho, V)$ of $G$ on a vector space $V$, we can construct the associated bundle $P_{(\rho,V)} := (P \times V)/\sim_\rho$, where $p \times v \sim_\rho p' \times v'$, iff $p' \times v' = p\, g \times \rho(g)^{-1} v$ for some $g \in G$.

# Generalities II

- Let us define the following category PrGBund:
  - An object in PrGBund is a principal bundle $G \hookrightarrow P \stackrel{\pi}{\to} M$ over an oriented and time-oriented globally hyperbolic Lorentzian manifold $M$.
  - A morphism between two objects $G \hookrightarrow P_1 \stackrel{\pi_1}{\to} M_1$ and $G \hookrightarrow P_2 \stackrel{\pi_2}{\to} M_2$ is a principal bundle map $f : P_1 \to P_2$, such that $\underline{f} : M_1 \to M_2$ is as in Man.

**Def:** (i) A locally gauge covariant QFT is a covariant functor $\mathfrak{A} :$ PrGBund $\to$ Alg$^{\mathrm{ni}}$, where Alg$^{\mathrm{ni}}$ is the category of unital $*$-algebras with possibly non-injective unital $*$-algebra homomorphisms as morphisms.

(ii) A locally gauge covariant QFT $\mathfrak{A}$ is causal, if whenever there are morphisms $f_1 : P_1 \to P$ and $f_2 : P_2 \to P$ such that $\underline{f_1}[M_1]$ and $\underline{f_2}[M_2]$ are causally disjoint in $M$, then

$$\Big[\mathfrak{A}(f_1)\big(\mathfrak{A}(G \hookrightarrow P_1 \stackrel{\pi_1}{\to} M_1)\big), \mathfrak{A}(f_2)\big(\mathfrak{A}(G \hookrightarrow P_2 \stackrel{\pi_2}{\to} M_2)\big)\Big] = \{0\} \ .$$

(iii) A locally gauge covariant QFT $\mathfrak{A}$ satisfies the time-slice axiom, if whenever $f : P_1 \to P_2$ is such that $\underline{f}[M_1]$ contains a Cauchy surface of $M_2$, then $\mathfrak{A}(f) : \mathfrak{A}(G \hookrightarrow P_1 \stackrel{\pi_1}{\to} M_1) \to \mathfrak{A}(G \hookrightarrow P_2 \stackrel{\pi_2}{\to} M_2)$ is an isomorphism.

**Rem:** This is not only the correct notion of covariance for dynamical gauge theories, but also for charged matter fields that live in associated bundles!

# Construction of the Maxwell functor

# Kinematics

Let $U(1) \hookrightarrow P \xrightarrow{\pi} M$ be an object in PrGBund with $G = U(1)$.

- A connection is an element $A \in \Omega^1(P, \mathfrak{g})$, such that
  - $A(X^\xi) = \xi$ for all $\xi \in \mathfrak{g}$ ("verticality condition")
  - $r_g^*(A) = \mathrm{ad}_{g^{-1}}(A)$ for all $g \in G$ ("adjoint $U(1)$-equivariance")

- With a little bit of math one shows that there is a 1-to-1 correspondence between connections $A$ and splittings $\lambda$ of the Atiyah sequence

$$0 \longrightarrow \mathrm{ad}(P) \xrightarrow{\;X^{\mathfrak{g}}\;} TP/G \underset{\lambda}{\overset{\pi_*}{\rightleftarrows}} TM \longrightarrow 0$$

- These splittings can be equivalently regarded as sections of the bundle of connections $\pi_{\mathcal{C}(P)} : \mathcal{C}(P) \to M$, that is an affine subbundle of $\mathrm{Hom}(TM, TP/G)$ with underlying vector bundle $\mathrm{Hom}(TM, \mathrm{ad}(P))$.

- The advantage of this formulation in terms of sections $\lambda$ of the bundle of connections instead of one-forms $A$ is that $\lambda$ is a field on $M$, while $A$ is a field on $P$.

$\longrightarrow$ Better control over the Cauchy problem, propagation and causality!

# Dynamics

- The curvature is an affine differential operator $F : \Gamma^\infty(\mathcal{C}(P)) \to \Omega^2(M)$, i.e., for all $\lambda \in \Gamma^\infty(\mathcal{C}(P))$ and $\eta \in \Omega^1(M)$, $F(\lambda + \eta) = F(\lambda) - \mathrm{d}\eta$.

- Composing with the co-differential $\delta = * \circ \mathrm{d} \circ *$ we define Maxwell's affine differential operator

$$\mathrm{MW} : \Gamma^\infty(\mathcal{C}(P)) \to \Omega^1(M) \ , \quad \lambda \mapsto \mathrm{MW}(\lambda) = \delta F(\lambda) \ ,$$

which has the linear part $\mathrm{MV}_V = -\delta \circ \mathrm{d} : \Omega^1(M) \to \Omega^1(M)$.

**NB:** The space of solutions $\mathrm{Sol} := \{\lambda \in \Gamma^\infty(\mathcal{C}(P)) : \mathrm{MW}(\lambda) = 0\}$ is an affine space over $\mathrm{Sol}_V := \{\eta \in \Gamma^\infty(\mathcal{C}(P)) : \delta \mathrm{d}\eta = 0\}$. It can be constructed explicitly from Cauchy data and gauge equivalence. Since we do not need this later, I will skip the details and refer to [Sanders et al. arXiv:1211.6420 [math-ph]] for a similar discussion.

# Observables and phasespace

For setting up a phasespace, we require basic observables, comparable to the smeared fields $\Phi(\varphi) = \int_M \Phi(x)\varphi(x)\,\mathrm{vol}_M$.

- Consider the dual bundle $\mathcal{C}(P)^\dagger$ of $\mathcal{C}(P)$, that is a vector bundle, and define the basic observables labelled by $\varphi \in \Gamma_0^\infty(\mathcal{C}(P)^\dagger)$

$$\mathcal{O}_\varphi : \Gamma^\infty(\mathcal{C}(P)) \to \mathbb{R}~,\quad \lambda \mapsto \mathcal{O}_\varphi(\lambda) = \int_M \varphi(\lambda)\,\mathrm{vol}_M$$

- Gauge invariance restricts to $\varphi$ satisfying $\int_M \varphi_V\big(\hat{f}^\ast(\mu_G)\big)\,\mathrm{vol}_M = 0$, for all $\hat{f} \in C^\infty(M,U(1))$.

- Taking the on-shell quotient $\mathcal{E}_{(P,M)} := \mathcal{E}^{\mathrm{inv}}/\mathrm{MW}^\ast[\Omega_0^1(M)]$, we can equip it with the pre-symplectic structure

$$\omega_{(P,M)} : \mathcal{E}_{(P,M)} \times \mathcal{E}_{(P,M)} \to \mathbb{R}~,\quad ([\varphi],[\psi]) \mapsto \omega_{(P,M)}([\varphi],[\psi]) = \int_M \varphi_V \wedge \ast(\Delta_M^\square(\psi_V))$$

**Thm:** The mapping $\mathfrak{PhSp} : \mathsf{PrGBund} \to \mathsf{PreSympl}$ specified by $\mathfrak{PhSp}(P,M) = \big(\mathcal{E}_{(P,M)}, \omega_{(P,M)}\big)$ and $\mathfrak{PhSp}(f) = f_\ast : \mathcal{E}_{(P_1,M_1)} \to \mathcal{E}_{(P_1,M_2)}$ is a covariant functor. Composing with the $\mathfrak{CCR}$-functor we obtain a locally gauge covariant QFT, which satisfies the causality property and time-slice axiom.

# Some non-trivial features and properties

- The linear part radical of the pre-symplectic structure $\omega_{(P,M)}$ is given by $\delta H_0^2(M)$ and therefore depends on the 2nd compact de Rham cohomology!
- This implies that $\mathfrak{A}$ : PrGBund $\to$ Alg$^{\mathrm{ni}}$ is not a functor to the full subcategory Alg, since e.g. for the embedding $\mathbb{R} \times S_2 \times (a,b) \hookrightarrow \mathbb{R} \times S_3$ we map all elements in $\delta H_0^2(M)$ to zero.
- An interpretation that the observables in the radical measure electric charges in "holes" appeared in [Sanders et al. arXiv:1211.6420 [math-ph]]. This looks OK, but I'm not completely convinced about this.
- Topological invariants of $P$: Let us look for simplicity at $M = \mathbb{R}^2 \times S_2$. Take any $\eta \in \Omega_0^2(M)$ of the form $\eta = \eta(x,y) * (\mathrm{d}x \wedge \mathrm{d}y)$ such that $\int_{\mathbb{R}^2} *(\eta) = \frac{1}{2\pi}$. Then the observable labelled by $F^*(\eta) \in \Gamma_0^\infty(\mathcal{C}(P)^\dagger)$ gives us the Euler class of $P$,

$$\mathcal{O}_{F^*(\eta)}(\lambda) = \int_M F^*(\eta)\big(\lambda\big)\mathrm{vol}_M = \int_M F(\lambda) \wedge *(\eta)$$
$$= \int_{S_2} F(\lambda) \underbrace{\int_{\mathbb{R}^2} *(\eta)}_{=\frac{1}{2\pi}} = \frac{1}{2\pi}\int_{S_2} F(\lambda) = e(P)$$

# Conclusions and outlook

# Conclusions

- I hope that I have convinced you that locally covariant QFT is a beautiful and powerful formulation of QFTs on curved spacetimes.
- Regarding a QFT as a covariant functor from a category of spacetimes to a category of algebras allows us to describe coherently QFTs on all spacetimes and also to introduce the principle of general covariance into QFT.
- This way of thinking is essential for renormalization on generic backgrounds and the formulation of the semiclassical Einstein equations.
- Gauge theories can be introduced by replacing the category of spacetimes by a category of principal bundles over spacetimes.
- The construction of the Maxwell functor points towards some nontrivial features of such theories, like the existence of a part of the center that depends on the cohomology groups, which leads to a nonlocal dependence in the observable algebras.
- It is thus important to further deepen our understanding of such models and to clarify their mathematical and physical properties.