Nonassociative geometry in quasi-Hopf representation categories

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Background and motivation

- In NC geometry, one studies modules V (left, right or bi) over algebras A.
- Structures of particular interest are connections $\nabla : V \to V \otimes_A \Omega^1(A)$ satisfying Leibniz rule $\nabla(v\,a) = \nabla(v)\,a + v \otimes \mathrm{d}a$. (Gauge theory!)
- **Problem:** For generic A and A-bimodules V, W there is no way to construct a connection on $V \otimes_A W$ from generic connections on V and W.

! Many standard examples in NC geometry (e.g. Moyal-Weyl plane, NC torus, Connes-Landi sphere) are very special NC algebras.

These algebras can be obtained by cocycle twist quantization and hence are commutative when regarded in the correct braided monoidal category!

- Recent developments in string theory (with R-fluxes) point us towards the relevance of nonassociative geometry [Blumenhagen,Lüst,...]:

$$[x^a, x^b] = i\,\mathsf{R}^{abc}\,p_c\ ,\quad [x^a, p_b] = i\,\delta^a_b\ ,\quad [p_a, p_b] = 0\ ,\quad [x^a, x^b, x^c] = i\,\mathsf{R}^{abc}\ .$$

- These algebras can be obtained by cochain twist quantization and hence are commutative and associative when regarded in the correct braided monoidal category! [Mylonas, Schupp, Szabo]
- **Goal:** Develop differential geometry on such algebras and their bimodules.

Quasi-Hopf representations and algebra objects

- Let k be commutative unital ring and H triangular quasi-Hopf algebra over k.
- Our constructions are within the $\mathbb{Z}$-graded representation category ${}^{H}\mathscr{M}$:
  - Objects: All bounded $\mathbb{Z}$-graded left H-modules $\triangleright : H \otimes V \to V$
  - Morphisms: All H-equivariant graded k-module maps $f : V \to W$
- Recall that ${}^{H}\mathscr{M}$ is a braided (even symmetric) monoidal category with
  - associator $\Phi : (v \otimes w) \otimes x \longmapsto (\phi^{(1)} \triangleright v) \otimes \big((\phi^{(2)} \triangleright w) \otimes (\phi^{(3)} \triangleright x)\big)$
  - braiding $\tau : v \otimes w \longmapsto (-1)^{|v|\,|w|}\ (R^{(2)} \triangleright w) \otimes (R^{(1)} \triangleright v)$
- Our spaces will be commutative algebra objects in ${}^{H}\mathscr{M}$, i.e. triples (A, μ, η) consisting of an ${}^{H}\mathscr{M}$-object A and two ${}^{H}\mathscr{M}$-morphisms $\mu : A \otimes A \to A$ (product) and $\eta : k \to A$ (unit), such that

$$\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{\mu \otimes \mathrm{id}} & A \otimes A \\
\Phi \downarrow & & \\
A \otimes (A \otimes A) & & \downarrow \mu \\
\mathrm{id} \otimes \mu \downarrow & & \\
A \otimes A & \xrightarrow[\mu]{} & A
\end{array}
\qquad
\begin{array}{ccccc}
k \otimes A & & & & A \otimes k \\
\eta \otimes \mathrm{id} \downarrow & \searrow^{\simeq} & & {}^{\simeq}\swarrow & \downarrow \mathrm{id} \otimes \eta \\
A \otimes A & \xrightarrow[\mu]{} & A & \xleftarrow[\mu]{} & A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\tau} & A \otimes A \\
& {}_{\mu}\searrow \quad \swarrow_{\mu} & \\
& A &
\end{array}$$

- We denote by ${}^{H}\mathscr{A}$ the category of algebra objects in ${}^{H}\mathscr{M}$ and by ${}^{H}\mathscr{A}^{\mathrm{com}}$ the full subcategory of commutative algebra objects.

Many examples via twisting G-manifolds

- Let G be a Lie group. Denote by G-Man the category of G-manifolds, i.e. manifolds with left G-action $\rho : G \times M \to M$.
- Let $U\mathfrak{g}$ be the universal enveloping Hopf algebra of the Lie algebra $\mathfrak{g}$ of G.

Prop: Differential forms on G-manifolds are functor $\Omega^\bullet(\,\cdot\,) : G\text{-Man}^{\mathrm{op}} \to {}^{U\mathfrak{g}}\mathscr{A}^{\mathrm{com}}$.

- Recall that a cochain twist of a quasi-Hopf algebra H is an invertible element $\mathcal{F} \in H \otimes H$ such that $(\epsilon \otimes \mathrm{id})\mathcal{F} = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{F}$.

Thm:
(i) Given a cochain twist $\mathcal{F}$ of a quasi-Hopf algebra H there is a new quasi-Hopf algebra $H_\mathcal{F}$ with coproduct given by $\Delta_\mathcal{F}(\,\cdot\,) = \mathcal{F}\,\Delta(\,\cdot\,)\,\mathcal{F}^{-1}$ and associator given by $\phi_\mathcal{F} := (1 \otimes \mathcal{F})\,(\mathrm{id} \otimes \Delta)(\mathcal{F})\,\phi\,(\Delta \otimes \mathrm{id})(\mathcal{F}^{-1})\,(\mathcal{F}^{-1} \otimes 1)$.
(ii) Cochain twisting Hopf algebras H in general produces quasi-Hopf algebras $H_\mathcal{F}$, unless one uses the special class of cocycle twists.

Prop: Given a cochain twist $\mathcal{F}$ of $U\mathfrak{g}$ there is a functor $\mathcal{F} : {}^{U\mathfrak{g}}\mathscr{A}^{\mathrm{com}} \to {}^{U\mathfrak{g}_\mathcal{F}}\mathscr{A}^{\mathrm{com}}$.

Ex:
1.) Use $G = \mathbb{T}^{2n}$ and Abelian twists $\mathcal{F} = \exp(\frac{\mathrm{i}}{2}\hbar\,\Theta^{ab} t_a \otimes t_b)$ to get the NC torus, Moyal-Weyl plane and Connes-Landi spheres as objects in ${}^{U\mathfrak{g}_\mathcal{F}}\mathscr{A}^{\mathrm{com}}$.
2.) Use $G = \mathrm{ISO}(2n)$ and the proper cochain twist of [Mylonas,Schupp,Szabo] $\mathcal{F} = \exp\big(-\frac{\mathrm{i}}{2}\hbar\,\big(\frac{1}{4}\,\mathsf{R}^{ijk}\,(m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\big)\big)$ to get the nonassociative algebras appearing in string theory as objects in ${}^{U\mathfrak{g}_\mathcal{F}}\mathscr{A}^{\mathrm{com}}$.

Internal hom-objects in ${}^H\mathscr{M}$

- The monoidal category ${}^H\mathscr{M}$ has internal homomorphisms, i.e. objects representing the functor $\mathrm{Hom}_{{}^H\mathscr{M}}(- \otimes V, W) : ({}^H\mathscr{M})^{\mathrm{op}} \longrightarrow$ Sets.
- The internal hom-objects $\hom(V,W)$ have a very explicit description:
  - underlying graded k-modules $\hom(V,W) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{l-m=n} \mathrm{Hom}_k(V_m, W_l)$
  - equipped with the adjoint H-action $h \triangleright L = (h_{(1)} \triangleright \,\cdot\,) \circ L \circ (S(h_{(2)}) \triangleright \,\cdot\,)$

Prop: There exist ${}^H\mathscr{M}$-morphisms

- internal evaluation: $\mathrm{ev} : \hom(V,W) \otimes V \longrightarrow W$
- internal composition: $\bullet : \hom(W,X) \otimes \hom(V,W) \longrightarrow \hom(V,X)$
- internal tensor: $\circledast : \hom(V,W) \otimes \hom(X,Y) \longrightarrow \hom(V\otimes X, W\otimes Y)$

satisfying "a bunch of" compatibility conditions, e.g. $\bullet$ is weakly-associative.

Cor:
(i) The internal endomorphisms $\mathrm{end}(V) = \hom(V,V)$ form a *noncommutative* algebra object in ${}^H\mathscr{M}$ with product given by $\bullet : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$ and unit by $\eta : k \to \mathrm{end}(V)\,,\ c \mapsto c\,(\beta \triangleright \,\cdot\,)$.

(ii) The internal endomorphisms $\mathrm{end}(V)$ form a Lie algebra object in ${}^H\mathscr{M}$ with Lie bracket $[\,\cdot\,,\,\cdot\,] := \bullet - \bullet \circ \tau : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$

Warming-up: Derivations of algebra objects

- Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$.

Lem: There exists an ${}^H\mathscr{A}$-morphism $\widehat{\mu} := \zeta(\mu) : A \to \mathrm{end}(A)$ given by "currying" the product map $\mu : A \otimes A \to A$.

- **Guiding principle:** Formulate algebraic properties via (co)equalisers in ${}^H\mathscr{M}$!
- Let's try this out for derivations:
  - Classically, a derivation X on an algebra A is a k-linear map $X : A \to A$, such that $X(ab) = X(a)b + aX(b)$ (Leibniz rule).
  - For an algebra object A in ${}^H\mathscr{M}$, the correct replacement of the Leibniz rule is given by the two parallel ${}^H\mathscr{M}$-morphisms

$$\mathrm{end}(A) \otimes A \underset{\widehat{\mu}\circ\mathrm{ev}}{\overset{[\,\cdot\,,\,\cdot\,]}{\rightrightarrows}} \mathrm{end}(A)$$

Def: We define the derivations on an algebra object A in ${}^H\mathscr{M}$ as the equalizer

$$\mathrm{der}(A) \longrightarrow \mathrm{end}(A) \underset{\zeta(\widehat{\mu}\circ\mathrm{ev})}{\overset{\zeta([\,\cdot\,,\,\cdot\,])}{\rightrightarrows}} \mathrm{hom}(A, \mathrm{end}(A))$$

Prop: $(\mathrm{der}(A), [\,\cdot\,,\,\cdot\,])$ is a Lie algebra subobject of $(\mathrm{end}(A), [\,\cdot\,,\,\cdot\,])$ in ${}^H\mathscr{M}$.

Connections on A-bimodule objects

- Let A be object in ${}^H\mathscr{A}$ together with H-invariant derivation $\mathrm{d} \in \mathrm{der}(A)$ which is of degree 1 and nilpotent $[\mathrm{d},\mathrm{d}] = 0$. (Differential calculus!)
- Consider symmetric A-bimodule objects in ${}^H\mathscr{M}$, denoted by ${}^H{}_A\mathscr{M}_A{}^{\mathrm{sym}}$. These are objects V in ${}^H\mathscr{M}$ together with ${}^H\mathscr{M}$-morphisms $l : A \otimes V \to V$ and $r : V \otimes A \to V$ (left and right actions) satisfying the usual bimodule conditions (given by ${}^H\mathscr{M}$-diagrams) and the symmetry condition $l \circ \tau = r$.

Ex: Cochain twisting of G-equivariant vector bundles!

Def: The k-connections on an object V in ${}^H{}_A\mathscr{M}_A{}^{\mathrm{sym}}$ are the equalizer in ${}^H\mathscr{M}$

$$k\text{-con}(V) \longrightarrow \mathrm{end}(V) \times k[1] \underset{\zeta(\nu)\circ\mathrm{pr}_{k[1]}}{\overset{\zeta([\,\cdot\,,\,\cdot\,])\circ\mathrm{pr}_{\mathrm{end}(V)}}{\rightrightarrows}} \mathrm{hom}(A, \mathrm{end}(V))$$

where $\nu : k[1] \otimes A \to \mathrm{end}(V)\,,\ (c,a) \mapsto c\,\widehat{l}(\mathrm{ev}(\mathrm{d} \otimes a))$.

Rem: k-connections are pairs $(\nabla, c) \in \mathrm{end}(V) \times k[1]$ satisfying (up to associator and R-matrices) the "**k**ontinuous Leibniz rule" $\nabla(v\,a) = \nabla(v)\,a + c\ v \otimes \mathrm{d}a$. Usual connections are special points $(\nabla, 1) \in k\text{-con}(V)$.

Prop: The curvature $R_{(\nabla,c)} := [\nabla, \nabla]$ of any k-connection is an internal endomorphism in ${}^H{}_A\mathscr{M}_A{}^{\mathrm{sym}}$.

Products of k-connections

- **Question:** Given two objects V, W in ${}^{H}{}_{A}\mathscr{M}_{A}{}^{\mathrm{sym}}$ and two k-connections (∇_V, c_V) and (∇_W, c_W). Can we form a k-connection on $V \otimes_A W$?
- Let us consider the fibred product

$$\begin{array}{ccc} k\text{-con}(V) \times_k k\text{-con}(W) & \dashrightarrow & k\text{-con}(W) \\ \vdots & & \downarrow \mathrm{pr}_{k[1]} \\ k\text{-con}(V) & \xrightarrow[\mathrm{pr}_{k[1]}]{} & k[1] \end{array}$$

Theorem (Main result (Barnes, AS, Szabo))

Let V, W be any two objects in ${}^{H}{}_{A}\mathscr{M}_{A}{}^{\mathrm{sym}}$. Then there is an ${}^{H}\mathscr{M}$-morphism (called the sum of k-connections)

$$\begin{aligned} \boxplus : k\text{-con}(V) \times_k k\text{-con}(W) &\longrightarrow k\text{-con}(V \otimes_A W) \; , \\ \big((\nabla_V, c), (\nabla_W, c)\big) &\longmapsto \Big(\nabla_V \circledbullet 1 + 1 \circledbullet \nabla_W, c\Big) \end{aligned}$$

Further aspects and an application

- The category ${}^{H}{}_{A}\mathscr{M}_{A}{}^{\mathrm{sym}}$ has internal-homs, denoted by $\hom_A(V,W)$. These are given by the equalizers

$$\hom_A(V,W) \longrightarrow \hom(V,W) \underset{0}{\overset{\zeta([\,\cdot\,,\,\cdot\,])}{\rightrightarrows}} \hom(A,\hom(V,W))$$

- There is an ${}^{H}\mathscr{M}$-morphism producing k-connections on internal-homs

$$\mathrm{ad}_\bullet : k\text{-con}(V) \ \times_k \ k\text{-con}(W) \ \longrightarrow \ k\text{-con}\big(\hom_A(V,W)\big) \ .$$

- **Application:**
  - Assume that you have given an object V in ${}^{H}{}_{A}\mathscr{M}_{A}{}^{\mathrm{sym}}$.
  - Then you can form the dual module $V^\vee := \hom_A(V,A)$ and consider the tensor algebra (over $\otimes_A$) generated by V and $V^\vee$ (classically, this is called (p,q)-tensor fields, at least in gravity textbooks...).
  - Given a connection ∇_V on V you can use our techniques to produce a connection on the whole tensor algebra generated by V and $V^\vee$.
  - This is important for example in (NC and NA) Riemannian geometry, where you start with a connection on, say, the tangent bundle and you want a connection on all (p,q)-tensor fields.

  $\Rightarrow$ Potential physical applications to NC and NA gravity.