Quantum field theories on Lorentzian manifolds

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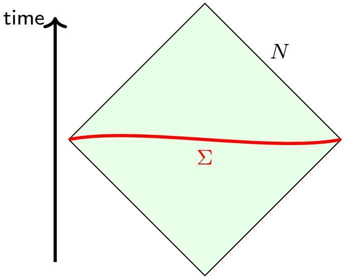
Based on a research program with Marco Benini, with contributions from

S. Bruinsma, S. Bunk, V. Carmona, C. Fewster, L. Giorgetti, A. Grant-Stuart, J. MacManus, G. Musante, M. Perin, J. Pridham, P. Safronov, U. Schreiber, R. Szabo and L. Woike.

Some background on Lorentzian spacetimes

- **Spacetime** $:=$ oriented and time-oriented globally hyperbolic Lorentzian manifold N



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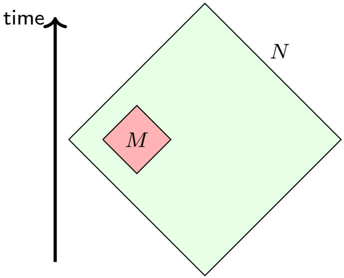

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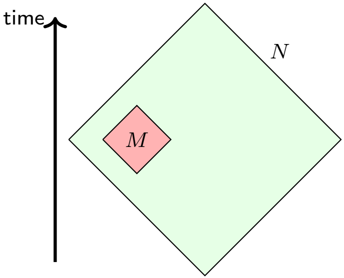

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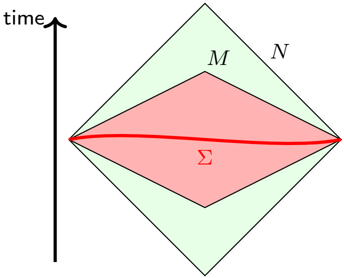

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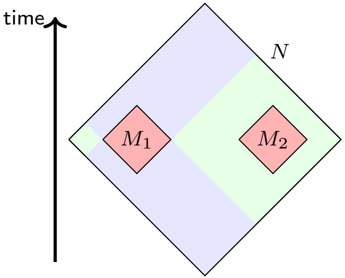

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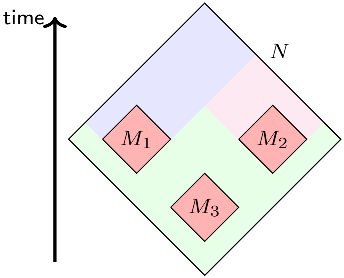

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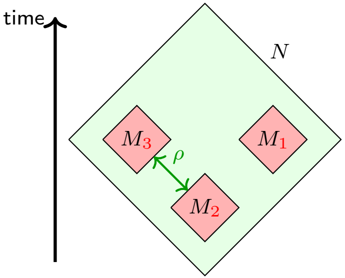

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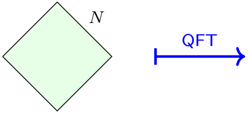

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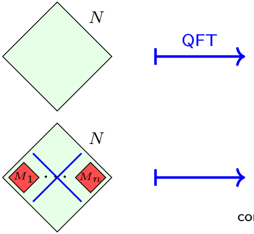

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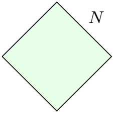

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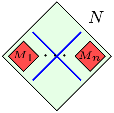

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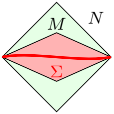

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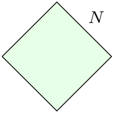

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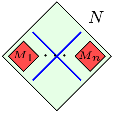

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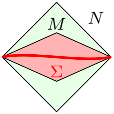

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◇ This is governed by the AQFT operad [Benini/AS/Woike, Benini/Carmona/AS]

$$\mathcal{O}_{(\mathbf{Loc}_m,\perp)}[\mathrm{Cauchy}^{-1}]^{\infty} \simeq \big(\mathcal{P}_{(\mathbf{Loc}_m,\perp)} \otimes \mathsf{uAs}\big)[\mathrm{Cauchy}^{-1}]^{\infty}$$

Classification in low dimensions (for target $\mathbf{T}$ = SM 1-category)

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \simeq \mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))} \quad ,$$

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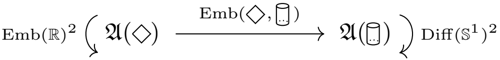

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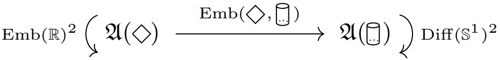

- *Open problem:* Higher dimensions? Some speculations later. . .

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\mathrm{char}\,\mathbb{K} = 0$)

- There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:
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Rem: Very different behavior to topological QFTs (via locally constant factorization algebras on $\mathbb{R}^m$) $\rightsquigarrow$ $\mathbb{E}_m$-algebras [Lurie, Ayala/Francis]

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

◇ *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

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Ex: Linear Yang-Mills theory [Benini/Bruinsma/AS]

$$\begin{array}{ccccccc}
\overset{(-1)}{\Omega^0_K(M)} & \xrightarrow{\ \mathrm{d}\ } & \overset{(0)}{\Omega^1_K(M)} & \xrightarrow{\ \delta\mathrm{d}\ } & \overset{(1)}{\Omega^1_K(M)} & \xrightarrow{\ \delta\ } & \overset{(2)}{\Omega^0_K(M)} \\
\downarrow\subseteq & \swarrow \delta G^{\pm}_{\square} & \downarrow\subseteq & \swarrow G^{\pm}_{\square} & \downarrow\subseteq & \swarrow \mathrm{d}G^{\pm}_{\square} & \downarrow\subseteq \\
\Omega^0_{J_M^{\pm}(K)}(M) & \xrightarrow[\ \mathrm{d}\ ]{} & \Omega^1_{J_M^{\pm}(K)}(M) & \xrightarrow[\ \delta\mathrm{d}\ ]{} & \Omega^1_{J_M^{\pm}(K)}(M) & \xrightarrow[\ \delta\ ]{} & \Omega^0_{J_M^{\pm}(K)}(M)
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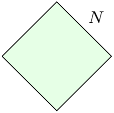

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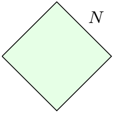

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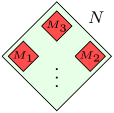

$$\longmapsto \qquad \underbrace{\bigotimes_{i=1}^{n} \mathfrak{F}(M_i) \longrightarrow \mathfrak{F}(N)}_{\text{time-ordered products}}$$

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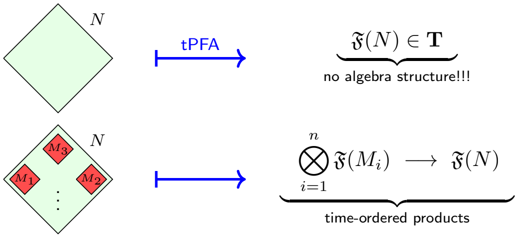

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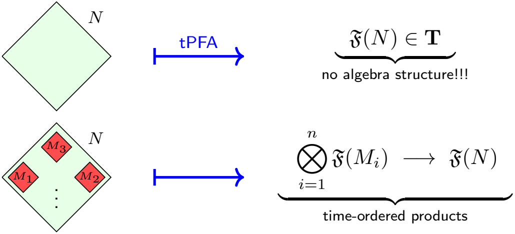

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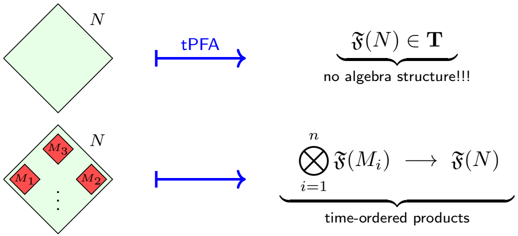

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◇ *Open problem:* Generalization to $\mathbf{T}$ = SM ∞-category, in particular $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$? In this case there are so far only example-based comparisons [Gwilliam/Rejzner, Benini/Musante/AS].

Comparison to functorial field theories (à la [Stolz/Teichner, . . .])

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◇ *Open problem:* What corresponds on the FFT side to the additional AQFT structure given by spatial locality? Is this related to extended field theories?

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Ex:

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(ii) Non-Abelian Yang-Mills theory on spatial lattices [Benini/Pridham/AS]