Homotopical algebraic quantum field theory

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Based on joint works with Marco Benini and different subsets of
{Urs Schreiber, Richard J. Szabo, Lukas Woike}

Outline

Background on AQFT

Basic idea [Haag,Kastler; Brunetti,Fredenhagen,Verch; . . .]

- Algebraic quantum field theory is an axiomatic approach to QFT on globally hyperbolic Lorentzian manifolds (= spacetimes)

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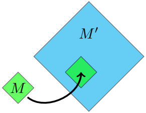

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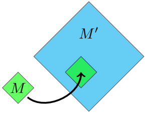

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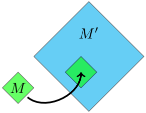

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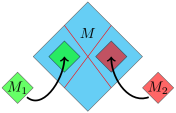

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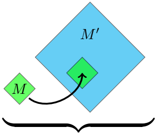

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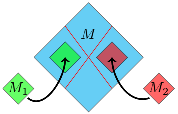

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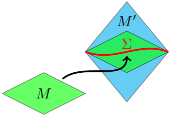
(ii) Time-Slice:



Cauchy morphism

$$\mathfrak{A}(M) \xrightarrow{\text{iso}} \mathfrak{A}(M')$$

The underlying algebraic structure

◇ **Input data:** (describing the type of QFTs)

- A triple $(\mathbf{C}, \perp, W)$ consisting of a category $\mathbf{C}$ with orthogonality relation $\perp \subseteq \mathrm{Mor}\,\mathbf{C}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{C}$ (symmetric & $\circ$-stable subset) and $W \subseteq \mathrm{Mor}\,\mathbf{C}$

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NB: The relevant categories are $\mathbf{QFT}(\mathbf{C}, \perp) := \mathbf{qft}(\mathbf{C}, \perp, \emptyset)$

Examples

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- $\mathbf{C} = \mathbf{Reg}_M$ the category of "nice" subsets $U \subseteq M$ in a ***fixed*** spacetime M
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Remark: Traditionally, the target category is chosen as $\mathbf{M} = \mathbf{Vec}$. We will later also consider model categories in order to do homotopy theory (i.e. gauge theory).

Operadic formulation

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- I will now show that for each orthogonal category $(\mathbf{C}, \perp)$ there exists a colored operad $\mathcal{O}_{(\mathbf{C},\perp)}$ such that

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     Homotopy theory of operads and their algebras is well understood! [Berger,Moerdijk; Hinich; Spitzweck; ...]

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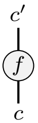

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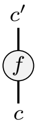

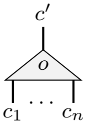

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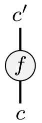

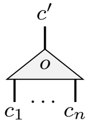

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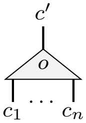

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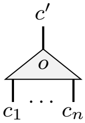

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(ii) operadic composition $\gamma : \mathcal{O}\big(\begin{smallmatrix} c \\ \underline{b} \end{smallmatrix}\big) \otimes \bigotimes_{i=1}^{n} \mathcal{O}\big(\begin{smallmatrix} b_i \\ \underline{a}_i \end{smallmatrix}\big) \longrightarrow \mathcal{O}\big(\begin{smallmatrix} c \\ (\underline{a}_1, \dots, \underline{a}_n) \end{smallmatrix}\big)$

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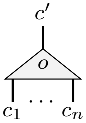

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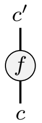
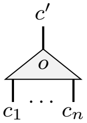
(iii) operadic unit $\mathbb{1} : I \longrightarrow \mathcal{O}\big(\substack{c \\ c}\big)$

Some background on colored operads

◇ Colored operads can be understood best by thinking of multicategories:

Category (1 in / 1 out) vs Colored operad (n in / 1 out)





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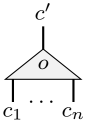

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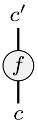

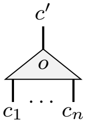

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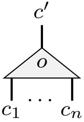
Ex: For $\mathfrak{C} = *$, we have the associative operad $\mathsf{As}(n) = \Sigma_n$.

. . . and their algebras

◇ $\mathcal{O}$-algebras can be understood best by thinking of representations:



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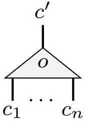

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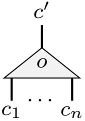

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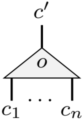

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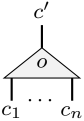

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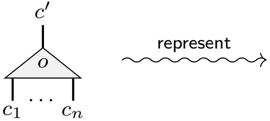

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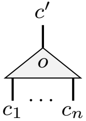

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Thm: For every colored operad morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$, the pullback functor $(f, \phi)^* : \mathbf{Alg}(\mathcal{P}) \to \mathbf{Alg}(\mathcal{O})$ has a left adjoint, i.e.

$$(f,\phi)_! \,:\, \mathbf{Alg}(\mathcal{O}) \rightleftarrows \mathbf{Alg}(\mathcal{P}) \,:\, (f,\phi)^*$$

The AQFT operads [Benini,AS,Woike]

- For $(\mathbf{C}, \perp)$ orthogonal category, define $\mathbf{C}_0$-colored operad $\mathcal{O}_{(\mathbf{C},\perp)}$ by

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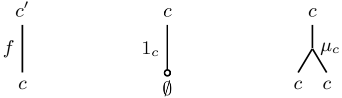

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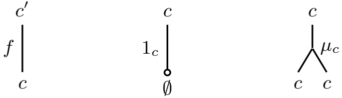

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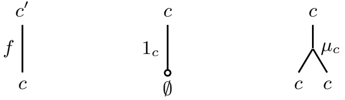

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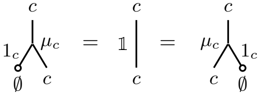

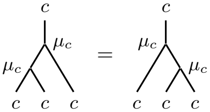

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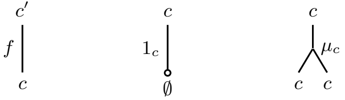

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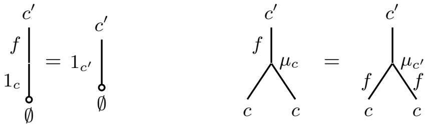

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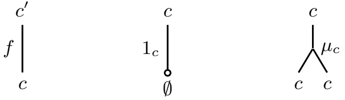

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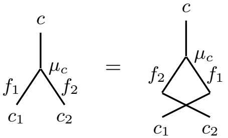

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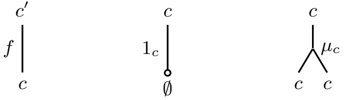

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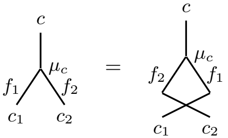

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Thm: The assignment $(\mathbf{C}, \perp) \mapsto \mathcal{O}_{(\mathbf{C},\perp)}$ is functorial on the category of orthogonal categories. There exists a natural isomorphism of categories

$$\mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \;\cong\; \mathbf{QFT}(\mathbf{C}, \perp)$$

Adjunctions between QFT categories

! Every orthogonal functor $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ defines operad map $\mathcal{O}_F : \mathcal{O}_{(\mathbf{C},\perp)} \to \mathcal{O}_{(\mathbf{C}',\perp')}$ and hence adjunction between algebra categories.

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The orthogonal functor $\mathrm{id}_{\mathbf{C}} : (\mathbf{C}, \emptyset) \to (\mathbf{C}, \perp)$ defines full reflective subcategory

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$\Rightarrow$ Structural result for the full subcategory $\mathbf{QFT}(\mathbf{C}, \perp) \subseteq \mathbf{Mon}(\mathbf{M})^{\mathbf{C}}$

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$\Rightarrow$ Time-slice axiom for $\mathfrak{A} \in \mathbf{QFT}(\mathbf{Loc}, \perp)$ is equivalent to $\eta_{\mathfrak{A}} : \mathfrak{A} \xrightarrow{\cong} \mathsf{U}\,\mathsf{T}\,\mathfrak{A}$

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Embedding $j : (\mathbf{Loc}_\diamond, j^*(\perp)) \to (\mathbf{Loc}, \perp)$ defines full **co**reflective subcategory

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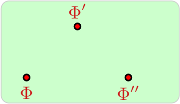
Rem: $\mathsf{ext} : \mathbf{QFT}(\mathbf{Loc}_\diamond, j^*(\perp)) \to \mathbf{QFT}(\mathbf{Loc}, \perp)$ is operadic refinement of Fredenhagen's universal algebra construction

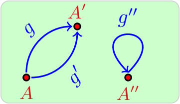
Homotopy theory of AQFTs

Higher structures in gauge theory

◇ **Gauge theory = higher spaces of fields**



"ordinary" field theory



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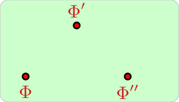

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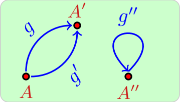

gauge theory

- Technically, these are described by (higher) stacks

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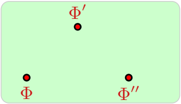

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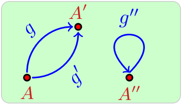

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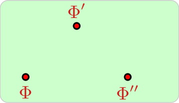

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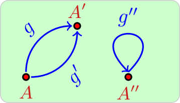

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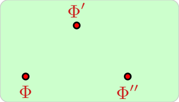

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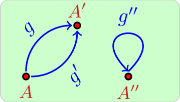

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Common feature of higher geometry and algebra

Higher spaces/algebras come with a notion of weak equivalences $X \stackrel{\sim}{\longrightarrow} Y$

$\Rightarrow$ Need for higher category theory or model category theory!

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   Derived extension functor $\mathbb{L}\mathrm{ext} : \mathbf{QFT}(\mathbf{Loc}_{\diamond}, j^*(\perp)) \longrightarrow \mathbf{QFT}(\mathbf{Loc}, \perp)$ is needed to obtain correct global gauge theory observables [Benini,AS,Szabo]

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- Let $\pi : (\mathbf{D}, \pi^*(\perp)) \to (\mathbf{C}, \perp)$ be (strictified) orthogonal category fibered in groupoids and consider $\mathbf{QFT}(\mathbf{D}, \pi^*(\perp))$

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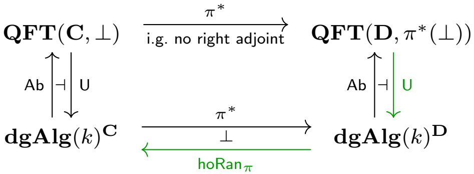

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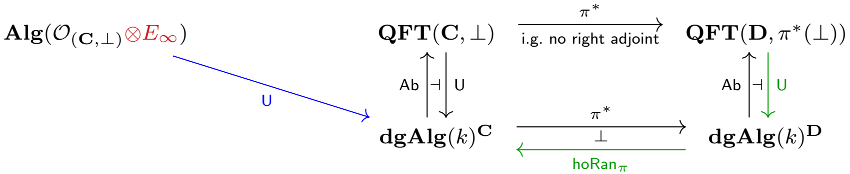

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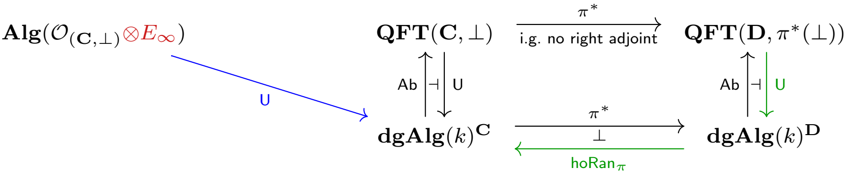

Thm: Using the typical Bousfield-Kan model

$$\mathsf{hoRan}_\pi\, \mathfrak{A}(c) \;=\; \int_{d \in \pi^{-1}(c)} \Big[N_*\big(B(\pi^{-1}(c) \downarrow d)\big), \mathfrak{A}(d) \Big] \quad ,$$

the functor $\mathsf{hoRan}_\pi\, \mathsf{U} : \mathbf{QFT}(\mathbf{D}, \pi^*(\perp)) \to \mathbf{dgAlg}(k)^{\mathbf{C}}$ admits a lift along $\mathsf{U} : \mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)} \otimes E_\infty) \to \mathbf{dgAlg}(k)^{\mathbf{C}}$.

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$\Rightarrow$ Interesting constructions, e.g. time-slicification and local-to-global

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× **Open problem:** Examples of quantum gauge theories, e.g. via deformation quantization of (derived) symplectic stacks [Calaque,Pantev,Toën,Vaquié,Vezzosi]