Interactions between algebra, geometry and quantum theory

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Abstract

Modern approaches to representation theory and quantum (field) theory encode algebraic structures in terms of m-dimensional geometric pictures, such as embeddings of multiple m-disks into a single bigger m-disk. This mini-course gives an elementary introduction to these subjects through the lens of factorization algebras. As concrete examples, we will see how quantum mechanics and also Lie algebra representations admit an interpretation in terms of 1-dimensional geometry. We will conclude with some comments on the much richer, but also more complicated, case of algebraic structures arising from $(m > 1)$-dimensional geometry.

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1 Background on algebras and representations

The aim of this section is to get familiar with some basic algebraic structures, in particular unital associative algebras and Lie algebras, and their representations. We will focus on algebraic structures that are defined on vector spaces (or later on cochain complexes of vector spaces) over a fixed field $\mathbb{K}$ of characteristic 0. In practice, $\mathbb{K}$ will be either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Before we can define the concept of an algebra, we have to recall the notion of *tensor products* of vector spaces.

Definition 1.1. A *tensor product* of two vector spaces V and W is a vector space $V \otimes W$ together with a bilinear map $V \times W \to V \otimes W\,,\;(v,w) \mapsto v \otimes w$ that satisfies the following universal property: For every bilinear map $f : V \times W \to Z$ to a vector space Z, there exists a unique linear map $\widehat{f} : V \otimes W \to Z$ such that the diagram

$$\begin{tikzcd}
V \times W \arrow[rr, "f"] \arrow[dr] & & Z \\
& V \otimes W \arrow[ur, dashed, "\exists !", "\widehat{f}"'] &
\end{tikzcd} \tag{1.1}$$

commutes, i.e. $f(v,w) = \widehat{f}(v \otimes w)$ for all $v \in V$ and $w \in W$.

Remark 1.2. Note that Definition 1.1 does *not* provide an explicit formula for the tensor product $V \otimes W$, but it defines it more abstractly by a so-called *universal property*. Defining the object of interest in terms of a universal property is typical in the context of category theory, see e.g. Leinster's book [Lei14] for an excellent introduction. The advantage is that the universal property makes it clear what the tensor product actually does: It turns bilinear maps $f : V \times W \to Z$ into linear maps $\widehat{f} : V \otimes W \to Z$ such that we have a bijection $\mathrm{Lin}(V \otimes W, Z) \cong \mathrm{BiLin}((V,W),Z)$ for all vector spaces Z. By abstract non-sense from category theory (see again [Lei14]), one shows that, provided it exists, the tensor product is unique up to canonical linear isomorphisms, which is why people often say *the* tensor product instead of the technically more correct, but awkward, phrase of *a* tensor product. To prove existence, note that there exists an explicit model for the tensor product that is given as follows: Denote by $\mathbb{K}\big(V \times W\big)$ the vector space that is spanned by all elements of the Cartesian product $V \times W$ and by $R \subseteq \mathbb{K}\big(V \times W\big)$ the sub-vector space spanned by the elements

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w) \quad , \tag{1.2a}$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2) \quad , \tag{1.2b}$$
$$(s\,v, w) - s\,(v, w) \quad , \tag{1.2c}$$
$$(v, s\,w) - s\,(v, w) \quad , \tag{1.2d}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $s \in \mathbb{K}$. Then the quotient vector space

$$V \otimes W \,:=\, \mathbb{K}\big(V \times W\big)/R \tag{1.3}$$

and the bilinear map $V \times W \to V \otimes W\,,\;(v,w) \mapsto [v,w]$ that assigns equivalence classes defines a tensor product. $\triangle$

The tensor product of vector spaces has very pleasant properties that can be proven by using the universal property.

Lemma 1.3. *There exist canonical linear isomorphisms*

$$\alpha_{V,W,Z} \,:\, (V \otimes W) \otimes Z \xrightarrow{\cong} V \otimes (W \otimes Z) \quad , \tag{1.4a}$$
$$\lambda_V \,:\, \mathbb{K} \otimes V \xrightarrow{\cong} V \quad , \tag{1.4b}$$
$$\rho_V \,:\, V \otimes \mathbb{K} \xrightarrow{\cong} V \quad , \tag{1.4c}$$
$$\tau_{V,W} \,:\, V \otimes W \xrightarrow{\cong} W \otimes V \quad . \tag{1.4d}$$

These isomorphisms endow the category of vector spaces over $\mathbb{K}$ and linear maps with the structure of a symmetric monoidal category, see e.g. [Ric20, Chapter 8] for the relevant definitions.

Remark 1.4. The isomorphisms α (called associator), λ (called left unitor) and ρ (called right unitor) are often suppressed from the notations. For example, one simply writes $V \otimes W \otimes Z$ for the tensor product of three vector spaces and implicitly understands that this can either mean $(V \otimes W) \otimes Z$ or the canonically isomorphic (via $\alpha_{V,W,Z}$) vector space $V \otimes (W \otimes Z)$. The reason why this is consistent is the content of Mac Lane's coherence theorem in category theory, see e.g. [ML98] for the details. △

After all these preparations, we are now finally ready to define a first important type of algebraic structure.

Definition 1.5. A *unital associative algebra* is a triple (A, μ, η) consisting of a vector space A, a linear map $\mu : A \otimes A \to A\,,~ a \otimes b \mapsto a\,b$ (called multiplication) and a linear map $\eta : \mathbb{K} \to A\,,~ s \mapsto s\,\mathbb{1}$ (called unit) that satisfy the following properties:

(1) *Associativity:* The diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \tag{1.5}$$

commutes, i.e. $(a\,b)\,c = a\,(b\,c) =: a\,b\,c$ for all $a, b, c \in A$.

(2) *Unitality:* The diagrams

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K} \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd} \tag{1.6}$$

commute, i.e. $\mathbb{1}\,a = a = a\,\mathbb{1}$ for all $a \in A$.

A unital associative algebra (A, μ, η) is called *commutative* if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{1.7}$$

commutes, i.e. $a\,b = b\,a$ for all $a, b \in A$.

Example 1.6. The simplest example of a unital associative algebra is $A = \mathbb{K}$ with multiplication $\mu : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}\,,~ s \otimes t \mapsto s\,t$ given by the multiplication of the field $\mathbb{K}$ and unit $\eta : \mathbb{K} \to \mathbb{K}\,,~ s \mapsto s$ given by the identity map. This algebra is commutative.

A more interesting family of examples is given by taking the vector space of $n \times n$-matrices $A = \mathrm{Mat}_{n\times n}(\mathbb{K})$ with multiplication $\mu : \mathrm{Mat}_{n\times n}(\mathbb{K}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{K}) \to \mathrm{Mat}_{n\times n}(\mathbb{K})$ given by matrix multiplication and unit $\eta : \mathbb{K} \to \mathrm{Mat}_{n\times n}(\mathbb{K})\,,~ s \mapsto s\,\mathbb{1}$ determined by the identity matrix $\mathbb{1}$. This algebra is not commutative (i.e. non-commutative) for $n \geq 2$. ▽

Example 1.7. Unital associative algebras also arise in quantum physics. For instance, the algebra of observables of a quantum particle in 1 dimension is the unital associative algebra A over $\mathbb{K} = \mathbb{C}$ that is generated by the position operator q and the momentum operator p, which are required to satisfy Heisenberg's commutation relation $q\,p - p\,q = \mathrm{i}\,\hbar\,\mathbb{1}$. One can write this algebra more formally as $A = \mathbb{C}[q, p]/I$, where $\mathbb{C}[q, p]$ denotes the free unital associative algebra generated by q and p, and $I = (q\,p - p\,q - \mathrm{i}\,\hbar\,\mathbb{1})$ denotes the two-sided ideal generated by the commutation relation. This algebra is non-commutative, which is an essential feature that distinguishes quantum physics from classical physics. ▽

A second type of algebraic structure that will be relevant in this mini-course is that of a Lie algebra.

Definition 1.8. A *Lie algebra* is a pair $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ consisting of a vector space $\mathfrak{g}$ and a linear map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\,,~ x\otimes y\mapsto [x,y]$ (called Lie bracket) that satisfies the following properties:

(1) *Antisymmetry:*

$$[x,y] = -[y,x] \quad , \tag{1.8}$$

for all $x,y\in\mathfrak{g}$.

(2) *Jacobi identity:*

$$[[x,y],z]+[[y,z],x]+[[z,x],y] \,=\, 0 \quad , \tag{1.9}$$

for all $x,y,z\in\mathfrak{g}$.

Example 1.9. A Lie algebra should be thought of as an object that describes "infinitesimal transformations". Let us illustrate this by considering a simple example that is related to the concept of spin in physics. Denote by

$$\mathsf{SU}(2) \,:=\, \Big\{U\in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,:\, U^\dagger = U^{-1} \text{ and } \det(U)=1\Big\} \tag{1.10}$$

the group of unitary 2×2-matrices with determinant 1, which is called the special unitary group of degree 2. The identity element of this group is the identity matrix $\mathbb{1}$ and the group operation is given by matrix multiplication. We are interested in transformations that are very close to the identity, which we model by matrix exponentials $U = e^{\lambda X} = \mathbb{1} + \lambda\, X + \mathcal{O}(\lambda^2)$ for λ a sufficiently small parameter such that the λ^2 terms can be neglected. (You can formalize this by working with nilpotent parameters satisfying $\lambda^2=0$.) For such U to be an element of $\mathsf{SU}(2)$, the exponent matrix $X\in\mathrm{Mat}_{2\times 2}(\mathbb{C})$ must be anti-Hermitian $X^\dagger=-X$ and trace-free $\mathrm{Tr}(X)=0$. The vector space

$$\mathfrak{su}(2) \,:=\, \Big\{X\in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,:\, X^\dagger = -X \text{ and } \mathrm{Tr}(X)=0\Big\} \tag{1.11}$$

of anti-Hermitian and trace-free 2×2-matrix thus describes the first-order infinitesimal neighborhood of the identity $\mathbb{1}$, i.e. it describes infinitesimal transformations. From the given group structure on $\mathsf{SU}(2)$ one can determine a Lie bracket on $\mathfrak{su}(2)$, which is given explicitly by the commutator for matrix multiplication

$$[X,Y] \,:=\, X\,Y - Y\,X \quad , \tag{1.12}$$

for all $X,Y\in\mathfrak{su}(2)$. A basis for the vector space underlying $\mathfrak{su}(2)$ is given by the three Pauli matrices (suitably normalized and rescaled by the imaginary unit $\mathrm{i}\in\mathbb{C}$ to be anti-Hermitian)

$$X_1 = -\frac{\mathrm{i}}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad , \quad X_2 = -\frac{\mathrm{i}}{2}\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix} \quad , \quad X_3 = -\frac{\mathrm{i}}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad . \tag{1.13}$$

The Lie bracket in this basis reads as

$$[X_a,X_b] \,=\, \epsilon_{abc}\,X_c \quad , \tag{1.14}$$

where ϵ_{abc} is the totally antisymmetric Levi-Civita symbol with $\epsilon_{123}=1$. Those of you who took a quantum physics module will recognize this as the Lie algebra that describes spin. $\triangledown$

Example 1.10. As a more mathematical example, we observe that every unital associative algebra (A,μ,η) has an underlying Lie algebra $(A,[\,\cdot\,,\,\cdot\,])$ with Lie bracket defined by the commutator $[a,b] := a\,b - b\,a$, for all $a,b\in A$. $\triangledown$

Construction 1.11. It turns out that there exists a construction in the reverse direction of Example 1.10, i.e. one can associate to every Lie algebra $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ a unital associative algebra that is denoted by $U\mathfrak{g}$ and called the *universal enveloping algebra*. For this one forms the free unital associative algebra over $\mathfrak{g}$, i.e.

$$T\mathfrak{g} \,:=\, \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \,:=\, \bigoplus_{n=0}^{\infty} \underbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}_{n\text{ times}} \tag{1.15}$$

with multiplication given by $\mu\big((x_1\otimes\cdots\otimes x_n)\otimes(x_{n+1}\otimes\cdots\otimes x_{n+m})\big) := x_1\otimes\cdots\otimes x_{n+m}$ and unit $\mathbb{1} := 1\in\mathbb{K}=\mathfrak{g}^{\otimes 0}\subseteq T\mathfrak{g}$, and defines

$$U\mathfrak{g} \,:=\, T\mathfrak{g}/I \tag{1.16}$$

as the quotient by the two-sided ideal $I=\big(x\otimes y-y\otimes x-[x,y]\,:\,\forall x,y\in\mathfrak{g}\big)$ that is determined by the Lie bracket. Note that there exists a Lie algebra morphism $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])\to(U\mathfrak{g},[\,\cdot\,,\,\cdot\,])\,,~x\mapsto x$ to the underlying Lie algebra from Example 1.10 of the universal enveloping algebra.

The universal enveloping algebra can also be defined more conceptually and abstractly through the following universal property: For every unital associative algebra (A,μ,η) and every Lie algebra morphism $h:(\mathfrak{g},[\,\cdot\,,\,\cdot\,])\to(A,[\,\cdot\,,\,\cdot\,])$ to its underlying Lie algebra, there exists a unique unital associative algebra morphism $\widehat{h}:(U\mathfrak{g},\mu,\eta)\to(A,\mu,\eta)$ such that the diagram of linear maps

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "h"] \arrow[dr] & A \\
& U\mathfrak{g} \arrow[u, dashed, "\exists!"', "\widehat{h}"]
\end{tikzcd} \tag{1.17}$$

commutes, i.e. we have a bijection $\mathrm{Hom}_{\mathrm{uAs}}\big((U\mathfrak{g},\mu,\eta),(A,\mu,\eta)\big)\cong\mathrm{Hom}_{\mathrm{Lie}}\big((\mathfrak{g},[\,\cdot\,,\,\cdot\,]),(A,[\,\cdot\,,\,\cdot\,])\big)$ between the set of unital associative algebra morphisms and the set of Lie algebra morphisms. ▷

To conclude this section, we introduce the concept of modules (also called representations) over a unital associative algebra and over a Lie algebra.

Definition 1.12. A *left module* (or representation) over a unital associative algebra (A,μ,η) is a pair (M,ℓ) consisting of a vector space M and a linear map $\ell:A\otimes M\to M\,,~a\otimes m\mapsto a\cdot m$ (called left action) that satisfies the following properties:

(1) *Compatibility with multiplication:* The diagram

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "\mathrm{id}\otimes\ell"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A\otimes M \arrow[d, "\ell"] \\
A\otimes M \arrow[r, "\ell"'] & M
\end{tikzcd} \tag{1.18}$$

commutes, i.e. $a\cdot(b\cdot m)=(a\,b)\cdot m$ for all $a,b\in A$ and $m\in M$.

(2) *Compatibility with unit:* The diagram

$$\begin{tikzcd}
\mathbb{K}\otimes M \arrow[rr, "\eta\otimes\mathrm{id}"] \arrow[dr, "\lambda_M"'] & & A\otimes M \arrow[dl, "\ell"] \\
& M &
\end{tikzcd} \tag{1.19}$$

commutes, i.e. $\mathbb{1}\cdot m=m$ for all $m\in M$.

Definition 1.13. A *left module* (or representation) over a Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ is a left module (M, ℓ) over the associated universal enveloping algebra $(U\mathfrak{g}, \mu, \eta)$ from Construction 1.11. This is equivalent to the datum of a pair $(M, \underline{\ell})$ consisting of a vector space M and a linear map $\underline{\ell} : \mathfrak{g} \otimes M \to M\,,\ x \otimes m \mapsto x \cdot m$ that satisfies

$$x \cdot (y \cdot m) - y \cdot (x \cdot m) \,=\, [x, y] \cdot m \quad , \tag{1.20}$$

for all $x, y \in \mathfrak{g}$ and $m \in M$.

Remark 1.14. There exists an analogous concept of *right modules*. It would be a good exercise for you to write down the relevant definition. △

Example 1.15. The following are simple examples of left modules over the algebras introduced in the examples above:

(i) The matrix algebra $A = \mathrm{Mat}_{n\times n}(\mathbb{K})$ from Example 1.6 has a left module given by $M = \mathbb{K}^n$ and matrix multiplication $\ell : \mathrm{Mat}_{n\times n}(\mathbb{K}) \otimes \mathbb{K}^n \to \mathbb{K}^n$.

(ii) Modulo functional analytical subtleties (that will not be discussed here, but can be controlled), the algebra of quantum observables from Example 1.7 has a left module given by the Hilbert space of wave functions $M = \mathcal{L}^2(\mathbb{R}) \ni \psi(x)$. Explicitly, the position operator acts as a multiplication operator $(q \cdot \psi)(x) = x\,\psi(x)$ and the momentum operator acts as a derivative operator $(p \cdot \psi)(x) = -\,\mathrm{i}\,\hbar\,\frac{d\psi(x)}{dx}$.

(iii) The Lie algebra $\mathfrak{su}(2)$ from Example 1.9 has a left module given by $M = \mathbb{C}^2$ and matrix multiplication $\underline{\ell} : \mathfrak{su}(2) \otimes \mathbb{C}^2 \to \mathbb{C}^2$. In physics, this representation is interpreted as a spin $\frac{1}{2}$ particle, and other values for the spin correspond to different representations.

It is important to emphasize that the classification of modules/representations is in general a difficult problem that is addressed in the field of representation theory. See e.g. [Hum72] for a good textbook on the representation theory of Lie algebras. ▽

2 The definition of a prefactorization algebra

In this section we introduce the concept of prefactorization algebras, which are more sophisticated algebraic structures that arise in quantum field theory [CG17, CG21], algebraic topology [AF15, AF20] and (higher-dimensional) representation theory [BZBJ18a, BZBJ18b]. Loosely speaking, prefactorization algebras are algebraic structures with a geometric origin, in the sense that they are determined by the shape of a manifold X. To avoid over-complicating things, we will typically consider the case where $X = \mathbb{R}^m$ is a Cartesian space of dimension $m \geq 1$, but if you are familiar with manifolds you can also take X to be a sphere, a torus, or any other manifold.

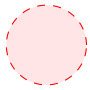
Let us fix an m-dimensional manifold X without boundary or, for simplicity, the Cartesian space $X = \mathbb{R}^m$.

Definition 2.1. An open subset $U \subseteq X$ is called an *m-disk* if it is diffeomorphic $U \cong \mathbb{R}^m$ to a Cartesian space. This means that there exists a smooth (i.e. infinitely often differentiable) map $f : U \to \mathbb{R}^m$ that admits a smooth inverse $f^{-1} : \mathbb{R}^m \to U$ such that $f^{-1} \circ f = \mathrm{id}$ and $f \circ f^{-1} = \mathrm{id}$.

Example 2.2. In low dimension, we recover some old friends:

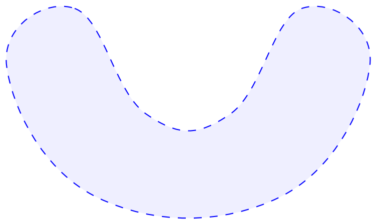
- For the 1-dimensional Cartesian space $X = \mathbb{R}^1$, a 1-disk is precisely an open interval $U = (a, b) \subseteq \mathbb{R}^1$.

- For the 2-dimensional Cartesian space $X = \mathbb{R}^2$, the open subset



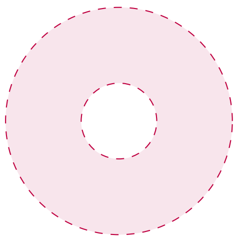
[that's a 2-disk] (2.1)

that looks like a disk is a 2-disk according to Definition 2.1, and so is the open subset



[that's also a 2-disk] (2.2)

that looks like a banana. To see a counterexample, note that an annulus



[that's **not** a 2-disk] (2.3)

is not a 2-disk.

- For the 3-dimensional Cartesian space $X = \mathbb{R}^3$, examples of 3-disks are given by open balls. Similarly to the bananas in 2 dimensions, open subsets that look like potatoes are also 3-disks.

What would be examples of 2-disks in the 2-dimensional sphere? $\triangledown$

We can now state the definition of a prefactorization algebra. At first sight, this looks quite involved, but we will illustrate and clarify this concept throughout this mini-course.

Definition 2.3. A *prefactorization algebra* $\mathfrak{F}$ on X consists of the following data:

(i) For every m-disk $U \subseteq X$, a vector space $\mathfrak{F}(U)$.

(ii) For every non-negative integer $n \geq 0$ and every tuple $((U_1, \dots, U_n), U)$ of m-disks in X such that $U_i \cap U_j = \emptyset$, for all $i \neq j$, are mutually disjoint and $U_i \subseteq U$, for all $i = 1, \dots, n$, a linear map (called structure map)

$$\mathfrak{F}^U_{(U_1,\dots,U_n)} \,:\, \bigotimes_{i=1}^n \mathfrak{F}(U_i) \,:=\, \mathfrak{F}(U_1) \otimes \cdots \otimes \mathfrak{F}(U_n) \,\longrightarrow\, \mathfrak{F}(U) \quad . \tag{2.4}$$

These data have to satisfy the following properties:

(1) *Compositionality:* For all families of tuples of mutually disjoint m-disks $((U_1, \dots, U_n), U)$ and $((U_{i1}, \dots, U_{ik_i}), U_i)$, for $i = 1, \dots, n$, the diagram

$$\begin{tikzcd}
\bigotimes\limits_{i=1}^n \bigotimes\limits_{j=1}^{k_i} \mathfrak{F}(U_{ij}) \arrow[rr, "\bigotimes_i \mathfrak{F}^{U_i}_{(U_{i1},\dots,U_{ik_i})}"] \arrow[dr, "\mathfrak{F}^U_{(U_{11},\dots,U_{nk_n})}"'] & & \bigotimes\limits_{i=1}^n \mathfrak{F}(U_i) \arrow[dl, "\mathfrak{F}^U_{(U_1,\dots,U_n)}"] \\
& \mathfrak{F}(U) &
\end{tikzcd} \tag{2.5}$$

commutes.

(2) *Identity:* For all m-disks $U \subseteq X$, the linear map $\mathfrak{F}_U^U = \mathrm{id} : \mathfrak{F}(U) \to \mathfrak{F}(U)$ is the identity.

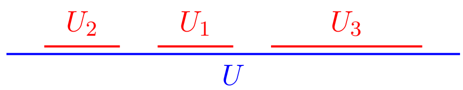
(3) *Permutation equivariance:* For all tuples of mutually disjoint m-disks $((U_1,\dots,U_n),U)$ and all permutations $\sigma \in \Sigma_n$ on n letters, the diagram

$$
\begin{tikzcd}
\bigotimes_{i=1}^{n} \mathfrak{F}(U_i) \arrow[rr, "\tau_\sigma", "\cong"'] \arrow[dr, "\mathfrak{F}^U_{(U_1,\dots,U_n)}"'] & & \bigotimes_{i=1}^{n} \mathfrak{F}(U_{\sigma(i)}) \arrow[dl, "\mathfrak{F}^U_{(U_{\sigma(1)},\dots,U_{\sigma(n)})}"] \\
& \mathfrak{F}(U) &
\end{tikzcd}
\tag{2.6}
$$

commutes, where $\tau_\sigma : \bigotimes_{i=1}^{n} \mathfrak{F}(U_i) \to \bigotimes_{i=1}^{n} \mathfrak{F}(U_{\sigma(i)})\,,~ a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$ is the permutation of elements defined by the linear isomorphism in (1.4d).

Remark 2.4. To better understand and appreciate Definition 2.3 of a prefactorization algebra $\mathfrak{F}$ on X, let us look more closely at this structure and add some clarifying comments. The first observation is that a prefactorization algebra consists of many vector spaces, while the simpler algebraic structures from Section 1 are defined on a single vector space. These vector spaces have a geometric meaning as they are assigned to m-disks in X and one should think of $\mathfrak{F}(U)$ as describing some "local data that lives in the m-disk $U \subseteq X$". In physics, one thinks of $\mathfrak{F}(U)$ as describing local observables that one can measure in the region $U \subseteq X$.

The structure maps from Definition 2.3 (ii) define something like n-to-1 multiplication operations, but again with a geometric meaning: One can only multiply local data from mutually disjoint m-disks $U_1,\dots,U_n \subseteq X$ and the result will be a local datum in the larger m-disk $U \subseteq X$ that includes all the U_i's. As a simple example, consider the following configuration of 1-disks in $X = \mathbb{R}^1$

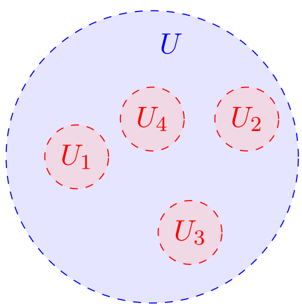


(2.7a)

and note that a prefactorization algebra assigns to this picture the structure map

$$
\mathfrak{F}^U_{(U_1,U_2,U_3)} ~:~ \mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \otimes \mathfrak{F}(U_3) ~\longrightarrow~ \mathfrak{F}(U) \quad . \tag{2.7b}
$$

Such pictures become more interesting in higher dimensions. For example, a prefactorization algebra on $X = \mathbb{R}^2$ assigns to the 2-disk configuration



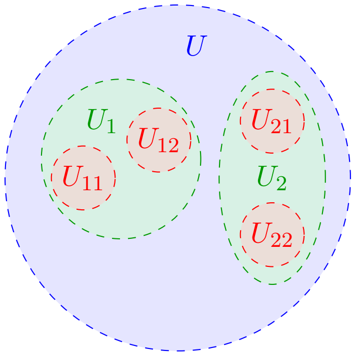
(2.8a)

the structure map

$$
\mathfrak{F}^U_{(U_1,U_2,U_3,U_4)} ~:~ \mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \otimes \mathfrak{F}(U_3) \otimes \mathfrak{F}(U_4) ~\longrightarrow~ \mathfrak{F}(U) \quad . \tag{2.8b}
$$

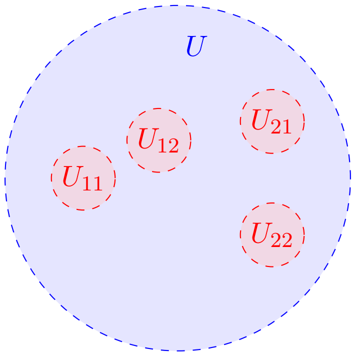
In physics, one thinks of these multiplications as combining a family of local observables in the U_i's to an observable that one can measure in the larger region U.

The properties listed in Definition 2.3 become very natural from this geometric point of view. Property (1) says that composing algebraically the n-to-1 multiplications coincides with composing geometrically the m-disk configurations. For example, consider an iterated 2-disk configuration in $X = \mathbb{R}^2$ of the form



(2.9)

that corresponds in the notation of Definition 2.3 to the following three tuples of mutually disjoint 2-disks $((U_1, U_2), U)$, $((U_{11}, U_{12}), U_1)$ and $((U_{21}, U_{22}), U_2)$. We can evaluate this picture in two different ways: The first option is to compose geometrically, i.e. forget the intermediate green 2-disks U_1 and U_2, which gives the configuration



(2.10)

of double-indexed 2-disks $((U_{11}, U_{12}, U_{21}, U_{22}), U)$ in U. The prefactorization algebra assigns to this picture the structure map

$$\mathfrak{F}^U_{(U_{11},U_{12},U_{21},U_{22})} \,:\, \mathfrak{F}(U_{11}) \otimes \mathfrak{F}(U_{12}) \otimes \mathfrak{F}(U_{21}) \otimes \mathfrak{F}(U_{22}) \longrightarrow \mathfrak{F}(U) \quad . \tag{2.11}$$

The second option is to assign to (2.9) via the prefactorization algebra the three structure maps

$$\begin{aligned} \mathfrak{F}^U_{(U_1,U_2)} &: \mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \longrightarrow \mathfrak{F}(U) \\ \mathfrak{F}^{U_1}_{(U_{11},U_{12})} &: \mathfrak{F}(U_{11}) \otimes \mathfrak{F}(U_{12}) \longrightarrow \mathfrak{F}(U_1) \\ \mathfrak{F}^{U_2}_{(U_{21},U_{22})} &: \mathfrak{F}(U_{21}) \otimes \mathfrak{F}(U_{22}) \longrightarrow \mathfrak{F}(U_2) \end{aligned} \tag{2.12}$$

and then compose algebraically

$$\mathfrak{F}^U_{(U_1,U_2)} \circ \Big(\mathfrak{F}^{U_1}_{(U_{11},U_{12})} \otimes \mathfrak{F}^{U_2}_{(U_{21},U_{22})} \Big) \,:\, \mathfrak{F}(U_{11}) \otimes \mathfrak{F}(U_{12}) \otimes \mathfrak{F}(U_{21}) \otimes \mathfrak{F}(U_{22}) \longrightarrow \mathfrak{F}(U) \quad . \tag{2.13}$$

Property (1) from Definition 2.3 then demands that the two linear maps in (2.11) and (2.13) are the same.

Properties (2) and (3) from Definition 2.3 are simpler: Property (2) states that associated with the trivial single m-disk inclusion $((U), U)$ is the identity map $\mathfrak{F}^U_U = \mathrm{id}$. Property (3) states that the multiplication operations $\mathfrak{F}^U_{(U_1,\dots,U_n)}$ do not depend on the ordering that one chooses to write down the tuple $(U_1, \dots, U_n)$ of mutually disjoint m-disks, which is reflected by the geometric pictures as in (2.8) where there is no canonical ordering of disks, besides the arbitrary numerical labels that we attach to the disks. △

There is an interesting subclass of prefactorization algebras that is related to topological quantum field theories. Loosely speaking, their main feature is that the vector space $\mathfrak{F}(U)$ that is assigned to an m-disk $U \subseteq X$ is insensitive to the "size" of the disk. This can be formalized as follows.

Definition 2.5. A prefactorization algebra $\mathfrak{F}$ on X is called *locally constant* if the structure map $\mathfrak{F}_U^{U'} : \mathfrak{F}(U) \xrightarrow{\cong} \mathfrak{F}(U')$ is a linear isomorphism, for every m-disk inclusion $U \subseteq U' \subseteq X$.

Remark 2.6. In their books [CG17, CG21], Costello and Gwilliam introduce also the concept of *factorization algebras*, which are prefactorization algebras that satisfy an additional local-to-global (i.e. descent) condition that is similar to that of a cosheaf. This condition will play no role in what we will discuss in this mini-course, hence we do not have to explain it in detail. △

3 Locally constant prefactorization algebras in 1 dimension

In this section we will describe in some detail the simplest case of locally constant prefactorization algebras on the 1-dimensional Cartesian space $X = \mathbb{R}^1$. We will see that these are related (in a precise way) to the more traditional algebraic concept of unital associative algebras from Section 1. This suggests that one should think of locally constant prefactorization algebras on higher-dimensional Cartesian spaces $X = \mathbb{R}^m$ as higher-dimensional versions of unital associative algebras. This suggestion was made precise in a theorem of Lurie, who identifies locally constant prefactorization algebras on $X = \mathbb{R}^m$ with the $\mathbb{E}_m$-algebras (a.k.a. little m-disk algebras) from algebraic topology. Details can be found in [LurHA], see also the excellent lecture notes [Tan20], but these topics go far beyond the scope of this mini-course.

The study of prefactorization algebras on $X = \mathbb{R}^1$ can be simplified by the following elementary observation.

Lemma 3.1. *Let $\mathfrak{F}$ be any (not necessarily locally constant) prefactorization algebra on $X = \mathbb{R}^1$. Then the collection of all structure maps from Definition 2.3 (ii) is completely determined by the sub-collection of structure maps $\mathfrak{F}^U_{(U_1,\dots,U_n)}$ that are associated with tuples $((U_1,\dots,U_n),U)$ which are ordered along $\mathbb{R}^1$, i.e. $U_1 < \dots < U_n$.*

Proof. Given any tuple $((U_1,\dots,U_n),U)$ of mutually disjoint 1-disks, there exists a unique permutation $\sigma \in \Sigma_n$ such that the permuted tuple $((U_{\sigma(1)},\dots,U_{\sigma(n)}),U)$ is ordered along $\mathbb{R}^1$. Using property (3) from Definition 2.3, we find that the structure map $\mathfrak{F}^U_{(U_1,\dots,U_n)}$ is determined by $\mathfrak{F}^U_{(U_{\sigma(1)},\dots,U_{\sigma(n)})}$, which completes the proof. □

To see how unital associative algebras are related to locally constant prefactorization algebras on $X = \mathbb{R}^1$, it is useful to understand first how one can pass from unital associative algebras to prefactorization algebras.

Construction 3.2. Let (A,μ,η) be a unital associative algebra, see Definition 1.5. Our goal is to build from this datum a prefactorization algebra $\mathfrak{F}_A$ on $X = \mathbb{R}^1$. To any 1-disk $U \subseteq \mathbb{R}^1$, we assign the underlying vector space

$$\mathfrak{F}_A(U) := A \tag{3.1}$$

of the algebra. Using Lemma 3.1, it suffices to define the structure maps for all *ordered* tuples $((U_1,\dots,U_n),U)$ of mutually disjoint 1-disks in $U \subseteq \mathbb{R}^1$, i.e. $U_1 < \dots < U_n$. We define

$$(\mathfrak{F}_A)^U_{(U_1,\dots,U_n)} := \mu_n : A^{\otimes n} \longrightarrow A \quad , \quad a_1 \otimes \dots \otimes a_n \longmapsto a_1 \cdots a_n \tag{3.2}$$

by multiplying the n elements $a_1,\dots,a_n \in A$ in the algebra (A,μ,η). For $n = 0$, this should be read as the unit $(\mathfrak{F}_A)^U_\emptyset := \eta : A^{\otimes 0} = \mathbb{K} \to A\,,~ s \mapsto s\,\mathbb{1}$. For an ordered tuple of length $n = 2$, we have that $(\mathfrak{F}_A)^U_{(U_1,U_2)} = \mu$ is the multiplication map of the algebra (A,μ,η). It is a good exercise for you to check that these structure maps satisfy the properties from Definition 2.3. Since $(\mathfrak{F}_A)^{U'}_U = \mathrm{id} : A \to A$ is the identity, for all $U \subseteq U' \subseteq X$, the prefactorization algebra $\mathfrak{F}_A$ on $X = \mathbb{R}^1$ is locally constant in the sense of Definition 2.5. $\triangleright$

Proposition 3.3. *Every locally constant prefactorization algebra $\mathfrak{F}$ on $X = \mathbb{R}^1$ is isomorphic to one of the form $\mathfrak{F}_A$ from Construction 3.2, for some unital associative algebra (A,μ,η).*

Proof. I will only sketch a proof and encourage you to fill in the details.

Using local constancy, we obtain for every 1-disk $U \subseteq \mathbb{R}^1$ a linear isomorphism

$$\mathfrak{F}^{\mathbb{R}^1}_U \,:\, \mathfrak{F}(U) \xrightarrow{\;\cong\;} \mathfrak{F}(\mathbb{R}^1) \tag{3.3}$$

that allows us to identify the vector space $\mathfrak{F}(U)$ with the vector space $\mathfrak{F}(\mathbb{R}^1)$ that is assigned to the whole real line $\mathbb{R}^1$, which we shall denote by

$$A := \mathfrak{F}(\mathbb{R}^1) \quad . \tag{3.4}$$

Given any *ordered* tuple $((U_1,\dots,U_n),U)$ of mutually disjoint 1-disks in $U \subseteq \mathbb{R}^1$, we define a linear map $A^{\otimes n} \to A$ by the commutative diagram

$$\begin{tikzcd}
A^{\otimes n} \arrow[rr, "\mu^U_{(U_1,\dots,U_n)}"] & & A \\
\bigotimes_{i=1}^n \mathfrak{F}(U_i) \arrow[u, "\bigotimes_i \mathfrak{F}^{\mathbb{R}^1}_{U_i}", "\cong"'] \arrow[rr, "\mathfrak{F}^U_{(U_1,\dots,U_n)}"'] & & \mathfrak{F}(U) \arrow[u, "\cong", "\mathfrak{F}^{\mathbb{R}^1}_U"']
\end{tikzcd} \quad . \tag{3.5}$$

Using property (1) from Definition 2.3, one finds that the linear map $\mu^U_{(U_1,\dots,U_n)}$ is independent of the choice of the big 1-disk U, hence we can set $U = \mathbb{R}^1$. Concerning dependence on the tuple $(U_1,\dots,U_n)$ of mutually disjoint 1-disks, we observe by using again property (1) that, given any $(U_1',\dots,U_n')$ such that $U_i' \subseteq U_i$ for all $i = 1,\dots,n$, we have

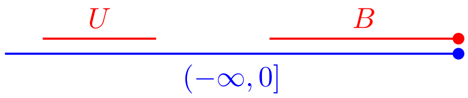
$$\mu^{\mathbb{R}^1}_{(U_1,\dots,U_n)} = \mu^{\mathbb{R}^1}_{(U_1',\dots,U_n')} \quad . \tag{3.6}$$

This implies that $\mu^{\mathbb{R}^1}_{(U_1,\dots,U_n)}$ depends only on the length of the *ordered* tuple of mutually disjoint 1-disks along $\mathbb{R}^1$, i.e. we can write $\mu_n := \mu^{\mathbb{R}^1}_{(U_1,\dots,U_n)}$. For low n, this defines a unit $\mu_0 : \mathbb{K} \to A\,,~ s \mapsto s\,\mathbb{1}$, gives the identity $\mu_1 = \mathrm{id} : A \to A\,,~ a \mapsto a$ and defines a multiplication $\mu_2 : A \otimes A \to A\,,~ a \otimes b \mapsto a\,b$. Using again property (1) from Definition 2.3, one shows that (A,μ_2,μ_0) defines a unital associative algebra and also that $\mu_n : A^{\otimes n} \to A\,,~ a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n$ is given by multiplying the n algebra elements. The desired isomorphism $\mathfrak{F} \xrightarrow{\cong} \mathfrak{F}_A$ of prefactorization algebras is defined by the components $\mathfrak{F}^{\mathbb{R}^1}_U : \mathfrak{F}(U) \to \mathfrak{F}_A(U) = A = \mathfrak{F}(\mathbb{R}^1)$, for all 1-disks $U \subseteq \mathbb{R}^1$. $\square$

Remark 3.4. Proposition 3.3 can be sharpened as follows: There exists an equivalence $\mathbf{PFA}^{\mathrm{l.c.}}_{\mathbb{R}^1} \simeq \mathbf{Alg}_{\mathrm{uAs}}$ between the category $\mathbf{PFA}^{\mathrm{l.c.}}_{\mathbb{R}^1}$ of locally constant prefactorization algebras on $X = \mathbb{R}^1$ and the category $\mathbf{Alg}_{\mathrm{uAs}}$ of unital associative algebras. This equivalence is exhibited by functors that can be defined using the techniques from Construction 3.2 and Proposition 3.3. $\triangle$

Example 3.5. Recalling Example 1.7 and Construction 1.11, we can use the results above to re-interpret the algebra of quantum observables $A = \mathbb{C}[q,p]/I$ of a 1-dimensional quantum particle and the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ as locally constant prefactorization algebras $\mathfrak{F}_A$ and $\mathfrak{F}_{U\mathfrak{g}}$ on $X = \mathbb{R}^1$. Hence, the theory of prefactorization algebras subsumes the traditional concepts from Section 1. We will see in Section 4 below how these examples can be constructed directly as prefactorization algebras by using more geometric techniques. $\triangledown$

You might now ask if it is possible to describe also the concept of modules (or representations) from Definition 1.12 in terms of prefactorization algebras. A left module over a unital associative algebra (A, μ, η) consists of a vector space M (in general different from A) and a left action $\ell : A \otimes M \to M$. This left-sidedness can be encoded geometrically by the half-line $X = (-\infty, 0]$



$$, \tag{3.7}$$

which comes with two different types of 1-disks: The first type are the usual open 1-disks $U = (a, b) \subseteq X$ in the interior, and the second type are half-open 1-disks $B = (a, 0] \subseteq X$ that include the boundary point 0.

One can now define in complete analogy to Definition 2.3 a concept of prefactorization algebras on $X = (-\infty, 0]$ that takes into account these two types of 1-disks. The result from Lemma 3.1 still holds true in this case. The correct generalization of the local constancy property from Definition 2.5 is to demand that $\mathfrak{F}_U^{U'} : \mathfrak{F}(U) \xrightarrow{\cong} \mathfrak{F}(U')$ and $\mathfrak{F}_B^{B'} : \mathfrak{F}(B) \xrightarrow{\cong} \mathfrak{F}(B')$ are linear isomorphisms, for all interior disk inclusions $U \subseteq U' \subseteq X$ and all boundary disk inclusions $B \subseteq B' \subseteq X$. Crucially, one should *not* demand that $\mathfrak{F}_U^B : \mathfrak{F}(U) \xrightarrow{\ncong} \mathfrak{F}(B)$ is an isomorphism for the inclusion $U \subseteq B \subseteq X$ of an interior disk into a boundary disk. (Note that there are no inclusions of the form $B \subseteq U \subseteq X$, hence there are no structure maps of the type $\mathfrak{F}_B^U$.) To understand how modules arise in the context of prefactorization algebras, we now generalize Construction 3.2 to the case of $X = (-\infty, 0]$.

Construction 3.6. Let (M, ℓ) be a left module over a unital associative algebra (A, μ, η), see Definition 1.12. Let us also choose any element $m_0 \in M$ in the module, whose role will become clearer below. Our goal is to build from this datum a prefactorization algebra $\mathfrak{F}_{(A,M,m_0)}$ on the half-line $X = (-\infty, 0]$. To any interior 1-disk $U = (a, b) \subseteq X$, we assign the underlying vector space

$$\mathfrak{F}_{(A,M,m_0)}(U) := A \tag{3.8}$$

of the algebra, and to any boundary 1-disk $B = (a, 0] \subseteq X$ we assign the underlying vector space

$$\mathfrak{F}_{(A,M,m_0)}(B) := M \tag{3.9}$$

of the module. Concerning the structure maps, we can assume all tuples of mutually disjoint disks to be ordered along $X = (-\infty, 0]$, and find the following three types of disk configurations

$$\big((U_1, \dots, U_n), U\big) \qquad \text{[all interior to interior]} \quad , \tag{3.10a}$$
$$\big((U_1, \dots, U_{n-1}, B_n), B\big) \qquad \text{[all but one interior to boundary]} \quad , \tag{3.10b}$$
$$\big((U_1, \dots, U_n), B\big) \qquad \text{[all interior to boundary]} \quad . \tag{3.10c}$$

To the first type, we assign as in Construction 3.2 the structure map

$$(\mathfrak{F}_{(A,M,m_0)})^U_{(U_1,\dots,U_n)} : A^{\otimes n} \longrightarrow A \quad , \quad a_1 \otimes \cdots \otimes a_n \longmapsto a_1 \cdots a_n \tag{3.11}$$

that multiplies n algebra elements. To the second type, we assign the structure map

$$\begin{aligned}(\mathfrak{F}_{(A,M,m_0)})^B_{(U_1,\dots,U_{n-1},B_n)} : A^{\otimes(n-1)} \otimes M &\longrightarrow M \quad , \\ a_1 \otimes \cdots \otimes a_{n-1} \otimes m &\longmapsto (a_1 \cdots a_{n-1}) \cdot m\end{aligned} \tag{3.12}$$

that is given by the left action. Using property (1) in Definition 1.12, one can also write this equivalently as $(a_1 \cdots a_{n-1}) \cdot m = a_1 \cdot (a_2 \cdot (\cdots (a_{n-1} \cdot m)))$. The structure maps associated with disk configurations of the third type make use of our chosen element $m_0 \in M$ in the module, and they read as

$$(\mathfrak{F}_{(A,M,m_0)})^B_{(U_1,\dots,U_n)} \,:\, A^{\otimes n} \longrightarrow M \quad , \quad a_1 \otimes \cdots \otimes a_n \longmapsto (a_1 \cdots a_n) \cdot m_0 \quad . \tag{3.13}$$

It is a good exercise for you to check that this defines a prefactorization algebra $\mathfrak{F}_{(A,M,m_0)}$ on $X = (-\infty, 0]$, which is locally constant in the sense explained in the text before this construction. ▷

The proof of the following result is similar to the one of Proposition 3.3 and it will be skipped for reasons of limited time.

Proposition 3.7. *Every locally constant prefactorization algebra $\mathfrak{F}$ on $X = (-\infty, 0]$ is isomorphic to one of the form $\mathfrak{F}_{(A,M,m_0)}$ from Construction 3.6, for some left module (M, ℓ) over a unital associative algebra (A, μ, η) and an element $m_0 \in M$.*

Example 3.8. Recalling Example 1.15, we make the following observations:

(i) Representations of the algebra of quantum observables of a quantum particle can be studied from the point of view of locally constant prefactorization algebras on $X = (-\infty, 0]$. The algebra of quantum observables A lives in the interior disks $U \subseteq X$ and the (Hilbert) space of wave functions M lives on the boundary disks $B \subseteq X$. It makes sense to interpret the distinguished element $m_0 \in M$ as the state in which one prepares the quantum system, e.g. the ground state.

(ii) Representations of a Lie algebra $\mathfrak{g}$ can be studied from the point of view of locally constant prefactorization algebras on $X = (-\infty, 0]$. In this case the Lie algebra lives, in the form of its universal enveloping algebra $U\mathfrak{g}$, in the interior disks $U \subseteq X$ and representations M live on the boundary disks $B \subseteq X$. The representations one gets in this way come with the additional datum of a distinguished element $m_0 \in M$.

If one does not wish to have a distinguished element $m_0 \in M$, one can modify the definition of prefactorization algebra on $X = (-\infty, 0]$ by discarding the structure maps that are associated with the third type of disk configurations in (3.10). ▽

4 Geometric examples of prefactorization algebras

So far, our examples of prefactorization algebras have been rather algebraic, in the sense that we take some algebraic input (e.g. a unital associative algebra) and assign to this a prefactorization algebra. The goal of this section is to illustrate geometric constructions of prefactorization algebras, focusing on our running examples given by quantum mechanics and universal enveloping algebras. To simplify the presentation, we will consider only the case of the real line $X = \mathbb{R}^1$, but it is important to emphasize that such geometric constructions generalize easily to higher-dimensional $X = \mathbb{R}^m$ and also to manifolds.

In order to construct examples of prefactorization algebras by geometric methods, one needs the concept of *cochain complexes* from homological algebra, see e.g. [Wei94]. A detailed introduction to this subject would go far beyond the scope of this mini-course, but fortunately it will be sufficient for us to understand the definitions of a cochain complex and of its cohomology.

Definition 4.1. A *cochain complex* (V, d) is a family $V = \{V^i\}_{i \in \mathbb{Z}}$ of vector spaces V^i, labeled by integers $i \in \mathbb{Z}$ (called degrees), together with a family $\mathrm{d} = \{\mathrm{d}^i : V^i \to V^{i+1}\}_{i \in \mathbb{Z}}$ of linear maps (called differential) that satisfy $\mathrm{d}^{i+1} \circ \mathrm{d}^i = 0$, for all $i \in \mathbb{Z}$. The *cohomology* of a cochain complex (V, d) is the family of vector spaces $H^\bullet(V, \mathrm{d}) = \{H^i(V, \mathrm{d})\}_{i \in \mathbb{Z}}$ defined by

$$H^i(V, \mathrm{d}) \,:=\, \frac{\mathrm{Ker}\big(\mathrm{d}^i : V^i \to V^{i+1}\big)}{\mathrm{Im}\big(\mathrm{d}^{i-1} : V^{i-1} \to V^i\big)} \quad , \tag{4.1}$$

for all $i \in \mathbb{Z}$.

Remark 4.2. The tensor product of vector spaces from Definition 1.1, as well as the isomorphisms from Lemma 1.3, generalize to the world of cochain complexes. See e.g. [BMS22, Section 2.1] for a collection of the relevant formulas. The most crucial difference is that the map (1.4d) gets so-called Koszul signs, i.e. $\tau_{V,W}(v \otimes w) = (-1)^{|v|\,|w|}\, w \otimes v$, for $v \in V^i$ and $w \in W^j$, where $|v| = i$ and $|w| = j$ denote the degrees. These signs make homological algebra a bit "messy" and cumbersome to work with in practice. $\triangle$

4.1 Quantum mechanics as a prefactorization algebra

Before we can talk about quantum mechanics, let us start by reviewing some basic concepts from classical mechanics. A classical particle is described by its trajectory $\Phi : U \to \mathsf{Space}\,,\ t \mapsto \Phi(t)$, where $U \subseteq \mathbb{R}^1$ is a 1-disk, interpreted as time interval, and Space is the space in which the particle moves. In the simplest case where space is also 1-dimensional, i.e. $\mathsf{Space} = \mathbb{R}^1$, we have that the trajectory is simply a real-valued a smooth function $\Phi \in C^\infty(U)$ on the time interval $U \subseteq \mathbb{R}^1$. (For physical space $\mathbb{R}^3$, the trajectory would be a smooth function taking values in 3-vectors.) The physical trajectory of the particle is selected by solving a differential equation, called equation of motion, that depends on the system one would like to model. For example, for the harmonic oscillator, the equation of motion is given by

$$P\Phi \,:=\, \Big(\tfrac{d^2}{dt^2} + m_0^2\Big)\,\Phi \,=\, 0 \quad , \tag{4.2}$$

where $m_0^2 > 0$ is some constant parameter. If you know about Lagrangians, note that this is the Euler-Lagrange equation for $L = \frac{1}{2}\big(\frac{d\Phi}{dt}\big)^2 - \frac{m_0^2}{2}\,\Phi^2$. We can encode the equation of motion in terms of the cochain complex

$$\mathfrak{E}(U) \,:=\, \Big(\overset{(0)}{C^\infty(U)} \xrightarrow{\ P\ } \overset{(1)}{C^\infty(U)}\Big) \tag{4.3}$$

that is non-trivial only in degrees 0 and 1, and whose differential is the equation of motion P. Note that the 0-th cohomology $H^0\mathfrak{E}(U) \cong \mathrm{Ker}\big(P : C^\infty(U) \to C^\infty(U)\big)$ is precisely the vector space of solutions of the equation of motion. (With some more efforts, using well-posedness of the initial value problem, one can prove that the first cohomology of $\mathfrak{E}(U)$ is trivial.)

Since quantum theory is formulated in terms of quantum observables, and not in terms of solutions of the equation of motion, our next step is to assign to (4.3) a suitable algebra of classical observables. The simplest (but good enough for our purposes) choice is to consider the polynomial algebra on (4.3), which we write as

$$\mathrm{Obs}^{\mathrm{cl}}(U) \,:=\, \mathrm{Sym}\Big(\overset{(-1)}{C^\infty_{\mathrm{c}}(U)} \xrightarrow{\ -P\ } \overset{(0)}{C^\infty_{\mathrm{c}}(U)}\Big) \quad , \tag{4.4}$$

where the subscript $_{\mathrm{c}}$ denotes smooth functions with compact support, i.e. functions that vanish outside of some *closed* interval $[a, b] \subseteq U \subseteq \mathbb{R}^1$. By definition, this is the free commutative algebra (valued in cochain complexes) with generators $\beta \in C^\infty_{\mathrm{c}}(U)$ in degree -1 and generators $\alpha \in C^\infty_{\mathrm{c}}(U)$ in degree 0. Due to the Koszul signs mentioned in Remark 4.2, we have that $\alpha\,\alpha' = \alpha'\,\alpha$, $\alpha\,\beta = \beta\,\alpha$ and $\beta\,\beta' = -\beta'\,\beta$. We denote a generic generator (i.e. of degree 0 or -1) by a symbol like φ and the Koszul sign by $\varphi\,\varphi' = (-1)^{|\varphi|\,|\varphi'|}\,\varphi'\,\varphi$. The differential d on (4.4) is given on the generators by minus the equation of motion operator $\mathrm{d}\beta := -P\beta$ and by $\mathrm{d}\alpha = 0$. If you are unfamiliar with the concept of free commutative algebras, it will probably help to know that a general element in $\mathrm{Obs}^{\mathrm{cl}}(U)$ is a linear combination of elements of the form $\varphi_1 \cdots \varphi_k$, for some $k \geq 0$. Our first observation is that classical observables form a prefactorization algebra.

Lemma 4.3. *Endow the collection of cochain complexes* $\mathrm{Obs}^{\mathrm{cl}}(U)$*, for all* 1*-disks* $U \subseteq \mathbb{R}^1$*, with the following structure maps*

$$(\mathrm{Obs}^{\mathrm{cl}})^U_{(U_1,\dots,U_n)} \,:\, \bigotimes_{i=1}^n \mathrm{Obs}^{\mathrm{cl}}(U_i) \,\longrightarrow\, \mathrm{Obs}^{\mathrm{cl}}(U) \quad,\quad \bigotimes_{i=1}^n \big(\varphi_{i1}\cdots\varphi_{ik_i}\big) \,\longmapsto\, \varphi_{11}\cdots\varphi_{nk_n} \quad, \tag{4.5}$$

for all tuples $((U_1,\dots,U_n),U)$ *of mutually disjoint* 1*-disks in* $U \subseteq \mathbb{R}^1$*. This defines a cochain complex-valued prefactorization algebra* $\mathrm{Obs}^{\mathrm{cl}}$ *on* $X = \mathbb{R}^1$*.*

Proof. The properties from Definition 2.3 (generalized to cochain complexes) follow immediately from the fact that $\mathrm{Obs}^{\mathrm{cl}}(U)$ is a commutative algebra in cochain complexes. □

The prefactorization algebra for the quantum particle is obtained by quantizing (i.e. deforming) the classical one from Lemma 4.3. The relevant technique is called Batalin-Vilkovisky (BV) quantization and it is explained in detail in [CG17], see also [BMS22] for an alternative presentation of the same topic. Without going too much into the details, let me say that the key idea is to modify the differential d on (4.4) to a new differential $\mathrm{d}^{\mathrm{q}} := \mathrm{d} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$ that takes into account quantum effects, which are parametrized by Planck's constant $\hbar$. The so-called BV Laplacian Δ_{BV} acts trivially on order 0 and 1 monomials in (4.4), i.e. $\Delta_{\mathrm{BV}}(\mathbb{1}) = \Delta_{\mathrm{BV}}(\alpha) = \Delta_{\mathrm{BV}}(\beta) = 0$, and it is defined on order 2 monomials by

$$\Delta_{\mathrm{BV}}(\alpha\,\alpha') \,=\, \Delta_{\mathrm{BV}}(\beta\,\beta') \,=\, 0 \quad,\qquad \Delta_{\mathrm{BV}}(\alpha\,\beta) \,=\, \int_U \alpha(t)\,\beta(t)\,\mathrm{d}t \quad. \tag{4.6}$$

One then extends Δ_{BV} as a second-order differential operator to all of (4.4), see e.g. [BMS22, Definition 2.4] for an explicit formula that we however will not need in this mini-course. An important feature of the BV Laplacian is that it is local, in the sense that $\Delta_{\mathrm{BV}}(\alpha\,\beta) = 0$ whenever the supports of α and β do not intersect, because in this case the integrand is zero. This allows one to prove the following result, see e.g. [CG17] or alternatively [BMS22, Section 4.1].

Proposition 4.4. *For every* 1*-disk* $U \subseteq \mathbb{R}^1$*, define the cochain complex* $\mathrm{Obs}^{\mathrm{q}}(U)$ *by modifying the differential* d *on the cochain complex* $\mathrm{Obs}^{\mathrm{cl}}(U)$ *from* (4.4) *to* $\mathrm{d}^{\mathrm{q}} := \mathrm{d} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$*. Endowing this family of cochain complexes with the same structure maps as in Lemma 4.3 defines a cochain complex-valued prefactorization algebra* $\mathrm{Obs}^{\mathrm{q}}$ *on* $X = \mathbb{R}^1$*.*

Remark 4.5. You will now ask for sure how the prefactorization algebra $\mathrm{Obs}^{\mathrm{q}}$ is related to the alternative one $\mathfrak{F}_A$ from Example 3.5 that describes a quantum particle in terms of its algebra of quantum observables $A = \mathbb{C}[q,p]/I$. One can show that the 0-th cohomology $H^0\mathrm{Obs}^{\mathrm{q}}$ of the cochain complex-valued prefactorization algebra from Proposition 4.4 is isomorphic to $\mathfrak{F}_A$, but constructing such isomorphism is not so easy. Instead of writing down a full proof, which can be worked out using the methods from [CG17] or alternatively [BMS22], I would like to explain you the main idea that allows you to understand roughly how this works. The essential ingredient is to use that the equation of motion (4.2) for the harmonic oscillator has a well-posed initial value problem. This allows us to identify (via a so-called quasi-isomorphism) the cochain complex $\mathfrak{E}(U)$ in (4.3) with the vector space $\mathbb{R}^2$ that describes the initial data $\big(\Phi(0), \frac{d\Phi(0)}{dt}\big) \in \mathbb{R}^2$. The 0-th cohomology of the classical observables $H^0\mathrm{Obs}^{\mathrm{cl}}(U) \cong \mathrm{Sym}[q,p]$ can then be identified with a polynomial algebra in two variables and one finds that the classical prefactorization algebra structure from Lemma 4.3 corresponds to multiplication in this polynomial algebra. The effect of the quantum differential $\mathrm{d}^{\mathrm{q}} := \mathrm{d} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$ entering the definition of $\mathrm{Obs}^{\mathrm{q}}$ is that this multiplication will be deformed, which when computed explicitly leads to a non-commutative multiplication that satisfies Heisenberg's commutation relation $q\,p - p\,q = \mathrm{i}\,\hbar\,\mathbb{1}$. △

4.2 Universal enveloping algebra as a prefactorization algebra

With a similar geometric construction as in the previous subsection, one can obtain a geometric model for the prefactorization algebra $\mathfrak{F}_{U\mathfrak{g}}$ from Example 3.5 that is determined by the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. I will now sketch this construction and refer you to [CG17, Section 3.4] or alternatively to [BSV23, Section 4] for the details.

To every 1-disk $U \subseteq X = \mathbb{R}^1$, we assign the cochain complex

$$\mathfrak{g}\otimes\Omega^\bullet(U) \,:=\, \Big(\, \overset{(0)}{\mathfrak{g}\otimes\Omega^0(U)} \xrightarrow{\;\mathrm{id}\otimes\mathrm{d}^{\mathrm{dR}}\;} \overset{(1)}{\mathfrak{g}\otimes\Omega^1(U)} \,\Big) \quad , \tag{4.7}$$

where $\Omega^0(U) := C^\infty(U)$ denotes the vector space of smooth functions (or 0-forms) and $\Omega^1(U) := C^\infty(U)\,\mathrm{d}t$ the vector space of smooth 1-forms. (A 1-form $\omega = g\,\mathrm{d}t \in \Omega^1(U)$ is the combination of a smooth function g with the 1-dimensional volume element $\mathrm{d}t$. Such objects arise in differential geometry and play a role in integration theory.) The differential is the so-called de Rham differential and it is defined by $\mathrm{d}^{\mathrm{dR}} f := \frac{df}{dt}\,\mathrm{d}t$. The Lie bracket $[\,\cdot\,,\cdot\,]$ of $\mathfrak{g}$ extends to a Lie bracket on the cochain complex (4.7) via

$$\big[x\otimes\alpha, y\otimes\beta\big] \,=\, [x,y]\otimes\alpha\wedge\beta \quad , \tag{4.8}$$

for all $x,y\in\mathfrak{g}$ and $\alpha,\beta\in\Omega^\bullet(U)$, where the wedge-product is defined by the multiplication of $C^\infty(U)$ and $\mathrm{d}t\wedge\mathrm{d}t = 0$. Explicitly,

$$f\wedge g \,=\, f\,g \quad , \quad f\wedge(g\,\mathrm{d}t) \,=\, f\,g\,\mathrm{d}t \,=\, (f\,\mathrm{d}t)\wedge g \quad , \quad (f\,\mathrm{d}t)\wedge(g\,\mathrm{d}t) = f\,g\,\mathrm{d}t\wedge\mathrm{d}t = 0 \quad , \tag{4.9}$$

for all $f,g\in C^\infty(U)$.

The analog of the classical observables (4.4) in the present example is given by

$$\mathfrak{F}^{\mathrm{cl}}(U) \,:=\, \mathrm{Sym}\Big(\, \overset{(-1)}{\mathfrak{g}\otimes\Omega^0_{\mathrm{c}}(U)} \xrightarrow{\;-\mathrm{id}\otimes\mathrm{d}^{\mathrm{dR}}\;} \overset{(0)}{\mathfrak{g}\otimes\Omega^1_{\mathrm{c}}(U)} \,\Big) \quad , \tag{4.10}$$

where the subscript $_{\mathrm{c}}$ denotes again compact support. In analogy to Lemma 4.3, we find

Lemma 4.6. *Endow the collection of cochain complexes $\mathfrak{F}^{\mathrm{cl}}(U)$, for all 1-disks $U\subseteq\mathbb{R}^1$, with the following structure maps*

$$(\mathfrak{F}^{\mathrm{cl}})^U_{(U_1,\dots,U_n)} \,:\, \bigotimes_{i=1}^n \mathfrak{F}^{\mathrm{cl}}(U_i) \,\longrightarrow\, \mathfrak{F}^{\mathrm{cl}}(U) \quad , \quad \bigotimes_{i=1}^n \big(\varphi_{i1}\cdots\varphi_{ik_i}\big) \,\longmapsto\, \varphi_{11}\cdots\varphi_{nk_n} \quad , \tag{4.11}$$

for all tuples $((U_1,\dots,U_n),U)$ of mutually disjoint 1-disks. This defines a cochain complex-valued prefactorization algebra $\mathfrak{F}^{\mathrm{cl}}$ on $X=\mathbb{R}^1$.

Concerning the quantization of this prefactorization algebra, we again deform the differential d of (4.10) to a new differential $\mathrm{d}^{\mathrm{q}} := \mathrm{d} + \mathrm{d}_{\mathrm{CE}}$. In contrast to the BV Laplacian from the previous subsection, we use in the present case the so-called *Chevalley-Eilenberg differential* d_{CE} that is determined by the given Lie bracket (4.8). Explicitly, we set $\mathrm{d}_{\mathrm{CE}}(\mathbb{1}) = \mathrm{d}_{\mathrm{CE}}(\varphi) = 0$ for the order 0 and 1 monomials, and

$$\mathrm{d}_{\mathrm{CE}}\big(\varphi\,\varphi'\big) \,:=\, (-1)^i\,[\varphi,\varphi'] \,=\, (-1)^i\,[x,x']\otimes\alpha\wedge\alpha' \tag{4.12}$$

for the order 2 monomials with $\varphi = x\otimes\alpha\in\mathfrak{g}\otimes\Omega^i_{\mathrm{c}}(U)$ and $\varphi' = x'\otimes\alpha'\in\mathfrak{g}\otimes\Omega^j_{\mathrm{c}}(U)$, with $i,j\in\{0,1\}$ denoting the form degree. One then extends d_{CE} to all of (4.10) according to the construction explained in [CG17, Section 3.4] or [BSV23, Section 4.1]. The following result is proven in [CG17, Proposition 3.4.1] and also in [BSV23, Proposition 4.7] via different methods. This proof is again not so easy and hence it will be skipped in this mini-course.

Proposition 4.7. *For every* 1*-disk* $U \subseteq \mathbb{R}^1$*, define the cochain complex* $\mathfrak{F}^{\mathrm{q}}(U)$ *by modifying the differential* d *on the cochain complex* $\mathfrak{F}^{\mathrm{cl}}(U)$ *from* (4.10) *to* $\mathrm{d}^{\mathrm{q}} := \mathrm{d} + \mathrm{d}_{\mathrm{CE}}$*. Endowing this family of cochain complexes with the same structure maps as in Lemma 4.6 defines a cochain complex-valued prefactorization algebra* $\mathfrak{F}^{\mathrm{q}}$ *on* $X = \mathbb{R}^1$*. The* 0*-th cohomology* $H^0\mathfrak{F}^{\mathrm{q}}$ *of this prefactorization algebra is isomorphic to the prefactorization algebra* $\mathfrak{F}_{U\mathfrak{g}}$ *of the universal enveloping algebra from Example 3.5.*

5 Outlook: Current research related to prefactorization algebras

The world of prefactorization algebras is much richer and more beautiful than what we could cover in this short mini-course. I hope that I was able to give you a useful taster, thereby sparking interest in this and related subjects. In case you would like to explore this subject further, I would like to point you to a selection of more advanced topics. Since prefactorization algebras are relevant in different fields, most notably in algebraic topology, representation theory and quantum field theory, I will give literature recommendations for each of these research directions.

Algebraic topology: The algebraic topologists' version of prefactorization algebras is called *factorization homology*. This was initiated by Lurie [LurHA] and developed further by Ayala and Francis [AF15], see also [AF20] for a good review and [Tan20] for excellent lecture notes. In the language of our notes, what factorization homology does is the following: 1.) It describes locally constant prefactorization algebras on Cartesian spaces $\mathbb{R}^m$, for any $m \geq 1$, substantially generalizing what we have done in Section 3 to higher dimensions and also to higher categories, e.g. their framework works well also for cochain complexes and not only for vector spaces. This leads to the famous $\mathbb{E}_m$-algebras, also known as little m-disk algebras from topology. 2.) After understanding these structures locally on $\mathbb{R}^m$, factorization homology constructs interesting invariants of m-dimensional manifolds by gluing local data on $\mathbb{R}^m$ via a local-to-global construction to global data on the manifold.

Representation theory: Factorization homology can be used to study and understand quantum groups and their representation theory from a geometric perspective. This line of research was pioneered by Jordan and collaborators, see e.g. [BZBJ18a, BZBJ18b] and also [GJS23]. Roughly speaking, the link between factorization homology and quantum groups is that $\mathbb{E}_2$-algebras with values in linear categories are braided monoidal categories, which includes the representation categories of quantum groups. The local-to-global constructions from [AF15] then can be used to develop new techniques and gain new insights in the study of quantum groups and their representation theory.

Quantum field theory: The original works of Costello and Gwilliam [CG17, CG21] already cover a lot of applications of prefactorization algebras to quantum field theory, including elegant constructions of non-interacting and perturbatively interacting examples as required in physics. The prefactorization algebra framework also gives an interesting new perspective on topological quantum field theories, see e.g. [ES19], and on vertex operator algebras in conformal quantum field theory, see e.g. [Wil17]. One of my personal interests lies in understanding the relationship between prefactorization algebras and algebraic quantum field theory, which is my original field of research. This relationship is by now well understood, thanks to the works [GR20, BPS20, BMS22], which leads to an interesting and fruitful cross-fertilization between different communities. Additionally, the combination of ideas from prefactorization algebras and algebraic quantum field theory leads to interesting new developments in the latter field, such as e.g. categorified variants of algebraic quantum field theories [BPSW21].

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6 Exercises

Exercise 6.1. Given two arbitrary vector spaces V and W, consider the vector space $V \otimes W := \mathbb{K}(V \times W)/R$ defined in (1.3) and the bilinear map $V \times W \to V \otimes W\,,\ (v,w) \mapsto v \otimes w := [v,w]$ that assigns equivalence classes. Prove that these data define a tensor product in the sense of Definition 1.1 by verifying the universal property.

Exercise 6.2. Work through the Construction 1.11 of the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ and look up (e.g. via Google) the concepts that you are not familiar with. For example, is it clear to you what is meant by a Lie algebra morphism or a unital associative algebra morphism? Once you have done this, verify the universal property of $U\mathfrak{g}$ that is stated in Construction 1.11.

Exercise 6.3. Show that the two alternative definitions of a left module over a Lie algebra $\mathfrak{g}$ from Definition 1.13 are indeed equivalent.

Exercise 6.4. Consider a locally constant prefactorization algebra $\mathfrak{F}$ on $X = \mathbb{R}^m$. Show that the vector spaces $\mathfrak{F}(U)$ and $\mathfrak{F}(U')$ that are associated with two arbitrary m-disks $U, U' \subseteq X$ are isomorphic.

Exercise 6.5. Verify that $\mathfrak{F}_A$ from Construction 3.2 satisfies the axioms of a prefactorization algebra on $X = \mathbb{R}^1$ from Definition 2.3.

Exercise 6.6. Try to generalize the Construction 3.2 of $\mathfrak{F}_A$ to prefactorization algebras on higher-dimensional Cartesian spaces $X = \mathbb{R}^m$, for $m \geq 2$. Show that this is possible provided that the input (A, μ, η) is a *commutative* unital associative algebra.

Exercise 6.7. Use well-posedness of the initial value problem to prove that the first cohomology of the cochain complex (4.3) is trivial.

Exercise 6.8. Convince yourself that the geometric construction of the cochain complex-valued prefactorization algebra $\mathrm{Obs}^{\mathrm{q}}$ from Subsection 4.1 generalizes to $X = \mathbb{R}^m$, for any dimension $m \geq 1$, if one takes instead of (4.2) the Laplace-type equation $P := \Delta + m_0^2 := \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + m_0^2$. The resulting higher-dimensional prefactorization algebra defines a simple example of a quantum field theory, namely the free scalar quantum field on $X = \mathbb{R}^m$.

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