Higher structures in algebraic quantum field theory

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Based on joint works w/ M. Benini, S. Bruinsma, M. Perin and L. Woike.

See also arXiv: 1903.02878 for a review.

Plan:

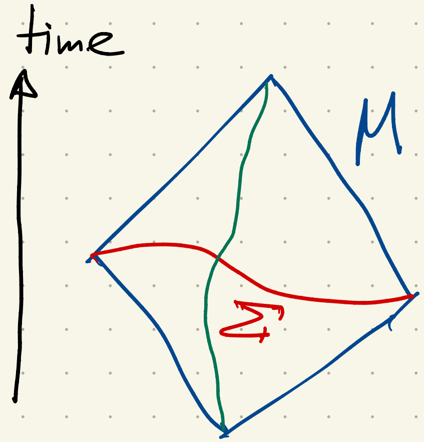
- Lecture I : Operads and universal constructions in AQFT
- Lecture II : Higher categories and quantum gauge theories
- Lecture III : Construction of simple examples by homological techniques

I. Operads and universal constructions in AQFT

AQFT studies QFTs on $\boxed{\text{Lorentzian manifolds}}$.

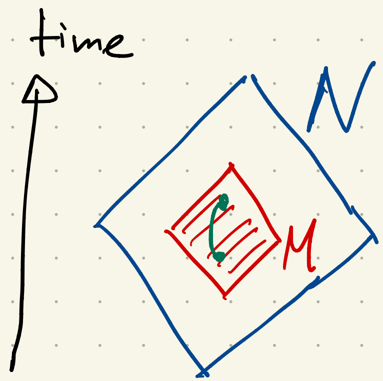
Def: • A Lorentzian manifold is a manifold M endowed w/ a metric of signature $(-++\cdots+)$.

• A Lorentzian manifold is called globally hyperbolic if it admits a Cauchy surface $\Sigma \subset M$, i.e. a codim 1 hypersurface that is intersected precisely once by each inextensible causal curve.

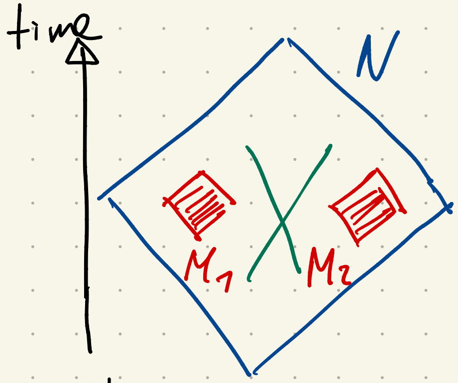


• Denote by **Loc** the category whose

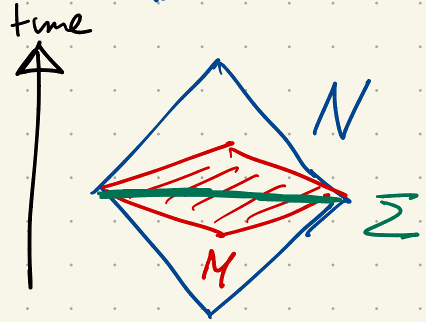
- objects are oriented and time-oriented g.h. Lorentzian mfds
- morphisms are orientation and time-orientation preserving isometric embeddings $f : M \to N$, such that $f(M) \subseteq N$ is open and causally convex.



- A pair $(M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2)$ of Loc-morphisms to a common target is called causally disjoint, written $f_1 \perp f_2$, if there exists no causal curve connecting $f_1(M_1)$ and $f_2(M_2)$.



- A Loc-morphism $f: M \to N$ is called a Cauchy morphism if $f(M) \subseteq N$ contains a Cauchy surface of N.



An AQFT links these Lorentzian geometric structures to algebraic structure on the quantum observables of a QFT.

For simplicity, I will neglect $(C)^*$-algebraic aspects throughout the whole lecture series and describe quantum observables of a QFT in terms of associative and unital algebras over a field $\mathbb{K}$ of char 0.

[See arXiv: 1802.09555 for $*$-structures in terms of involutive categories.]

Def: (Brunetti-Fredenhagen-Verch)

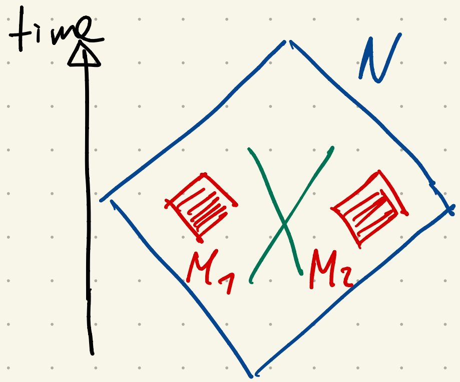
An AQFT is a functor $\mathfrak{A}: \mathbf{Loc} \longrightarrow \mathbf{Alg}_{\mathbb{K}}$ to the category of associative and unital $\mathbb{K}$-algebras that satisfies the following PROPERTIES:

i) Einstein causality: For every causally disjoint $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$, the induced commutator

$$[\cdot,\cdot]_{\mathfrak{A}(N)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N)$$

is zero.

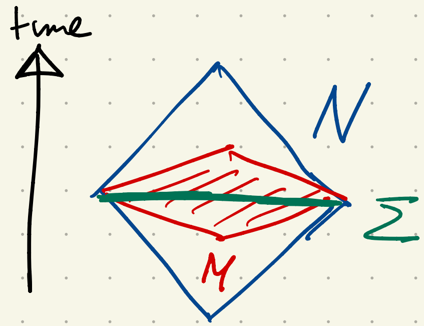
("Spacelike separated observables commute")



ii) Time-slice axiom: For every Cauchy morphism $f: M \to N$,

$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\cong} \mathfrak{A}(N)$ is an $\mathbf{Alg}_{\mathbb{K}}$-isomorphism.

("Time evolution")



PROPERTY vs STRUCTURE

Even though defining $\mathbf{AQFT} \subseteq \mathbf{Fun}(\mathbf{Loc}, \mathbf{Alg}_{\mathbb{K}})$ as the full subcategory of functors $\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathbf{Alg}_{\mathbb{K}}$ satisfying the PROPERTIES i) and ii) gives the correct definition, there are reasons why it is better to encode i) and ii) as STRUCTURE:

1.) PROPERTIES may be violated by universal constructions (e.g. colimits, Kan extensions, ...), while STRUCTURE is preserved.

2.) In higher category theory, commutativity (as in Einstein causality) is <u>not</u> a PROPERTY but an additional STRUCTURE. [Buzzword: E_n-algebras]

The time-slice axiom is easily implemented as STRUCTURE by localization $\mathbf{Loc} \longrightarrow \mathbf{Loc}[\mathrm{Cauchy}^{-1}]$ of categories.

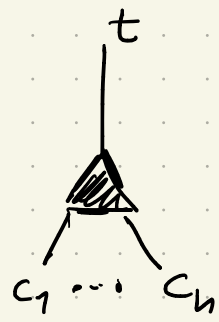
But Einstein causality is more complicated because it involves binary operations.

$\rightsquigarrow$ Use techniques from operad theory !

Def: A colored operad (a.k.a. multicategory) $\mathcal{O}$ consist of the following data:

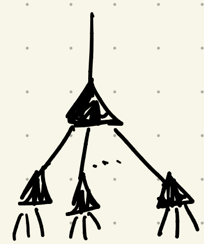
(1) A collection of objects $a, b, c, \ldots$

(2) For each tuple of objects $(\underline{c}, t) := ((c_1 \cdots c_n), t)$,
a set $\mathcal{O}\binom{t}{\underline{c}}$ of operations from $\underline{c}$ to t.



n-to-1 operations

(3) Composition maps $\gamma : \mathcal{O}\binom{t}{\underline{a}} \times \prod_{i=1}^{n} \mathcal{O}\binom{a_i}{\underline{b}_i} \longrightarrow \mathcal{O}\binom{t}{(\underline{b}_1 \cdots \underline{b}_n)}$



(4) Unit elements $1\!\!1 \in \mathcal{O}\binom{t}{t}$

(5) Permutation actions $\mathcal{O}(\sigma) : \mathcal{O}\binom{t}{\underline{c}} \longrightarrow \mathcal{O}\binom{t}{\underline{c}\sigma}$

satisfying suitable associativity, unitality and equivariance axioms.

Def: An $\mathcal{O}$-algebra A with values in a symmetric monoidal category $(\mathbf{T}, \otimes, I)$ consists of the following data:

(1) For each $c \in \mathcal{O}$, an object $A(c) \in \mathbf{T}$

(2) For each $\psi \in \mathcal{O}\binom{t}{\underline{c}}$, a $\mathbf{T}$-morphism $A(\psi) : \bigotimes_{i=1}^{n} A(c_i) \longrightarrow A(t)$

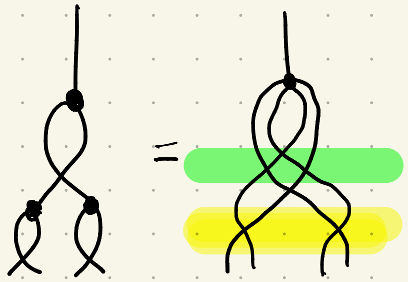
such that the assignment in (2) is compatible w/ compositions, units and permutation actions.

Example: The associative operad As is given by the following data:

(1) A single object $*$

(2) $As(n) := As\left(\begin{matrix} * \\ \underbrace{(* \cdots *)}_{n\text{-times}} \end{matrix}\right) = \Sigma_n$, the permutations on n letters

(3) $\gamma(G,(G_1,\dots,G_n)) = \underbrace{G\langle k_{G^{-1}(1)} \cdots k_{G^{-1}(n)}\rangle}_{\text{block permutation}} \cdot \underbrace{(G_1 \oplus \cdots \oplus G_n)}_{\text{sum permutation}}$



(4) $1\!\!1 = e_1 \in \Sigma_1$, the identity element

(5) $\Theta(G)(G') = G' \cdot G$, via group operation.

An As-algebra with values in $(Vec_{\mathbb{K}}, \otimes, \mathbb{K})$ is precisely an associative and unital $\mathbb{K}$-algebra with product $\mu := A(e_2) : A \otimes A \longrightarrow A$, $a_1 \otimes a_2 \longmapsto a_1 \cdot a_2$ and unit $\eta := A(e_0) : \mathbb{K} \longrightarrow A$, $\lambda \longmapsto \lambda \cdot 1$

For a general $G \in As(n) = \Sigma_n$, we have that

$$A(G) : A^{\otimes n} \longrightarrow A,$$
$$a_1 \otimes \cdots \otimes a_n \longmapsto a_{G^{-1}(1)} \cdot \cdots \cdot a_{G^{-1}(n)}$$

multiplication of n elements in the order dictated by $G \in \Sigma_n$.

The next example is relevant for QFT.

<u>Def</u>: An <u>orthogonal category</u> is a pair $\overline{\mathcal{C}} := (\mathcal{C}, \perp)$ consisting of a small category $\mathcal{C}$ and a subset $\perp \subseteq \operatorname{Mor}\mathcal{C} \,{}_t\!\times_t \operatorname{Mor}\mathcal{C}$ of the set of pairs of morphisms to a common target, satisfying:

i) Symmetry: If $(f_1, f_2) \in \perp$, then $(f_2, f_1) \in \perp$.

ii) $\circ$-stability: If $(f_1, f_2) \in \perp$, then $(g f_1 h_1, g f_2 h_2) \in \perp$ for all composable g, h_1 and h_2.

<u>Example</u>: The <u>prefactorization operad</u> $P_{\overline{\mathcal{C}}}$ associated w/ an orthogonal category $\overline{\mathcal{C}} = (\mathcal{C}, \perp)$ is given by the following data:

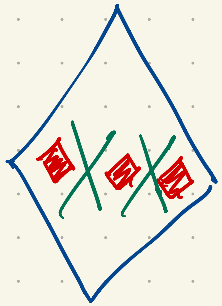
(1) Objects are the objects of $\mathcal{C}$

(2) $P_{\overline{\mathcal{C}}}\binom{t}{\underline{c}} = \left\{ \underline{f} = (f_1 \cdots f_n) \in \prod_{i=1}^{n} \operatorname{Hom}_{\mathcal{C}}(c_i, t) \;:\; f_i \perp f_j \;\forall i \neq j \right\}$

(3) $\gamma(\underline{f}, (\underline{g}_1 \cdots \underline{g}_n)) = (f_1 g_{11}, \cdots, f_n g_{n k_n})$

(4) $\mathbb{1} = \mathrm{id}_t \in P_{\overline{\mathcal{C}}}\binom{t}{t}$

(5) $P_{\overline{\mathcal{C}}}(\sigma)(f_1 \cdots f_n) = (f_{\sigma(1)} \cdots f_{\sigma(n)})$



Rem: For $\overline{\mathbf{C}} = \mathbf{Open}(M)$, the category of open subsets of a mnf M, and $\perp$ defined by disjointness, i.e. $(U_1 \subseteq V) \perp (U_2 \subseteq V)$ iff $U_1 \cap U_2 = \emptyset$, $\mathcal{P}_{\overline{\mathbf{C}}}$ is the operad used for factorization algebras à la Costello / Gwilliam. $\triangle$

In AQFT, we consider $\overline{\mathbf{Loc}} = (\mathbf{Loc}, \perp)$ with $\perp$ given by causal disjointness and also its localization $\overline{\mathbf{Loc}}[\mathrm{Cauchy}^{-1}]$ at all Cauchy morphisms.

Theorem: There exist canonical equivalences of categories

$$\{\text{AQFTs w/out time-slice}\} \simeq \{\mathcal{P}_{\overline{\mathbf{Loc}}}\text{-algebras in } \mathbf{Alg}_{\mathbb{K}}\}$$

$$\simeq \{\underbrace{\mathcal{P}_{\overline{\mathbf{Loc}}} \otimes_{\mathrm{BV}}}_{\text{Boardman-Vogt tensor product}} \mathsf{As}\text{-algebras in } \mathbf{Vec}_{\mathbb{K}}\}$$

and

$$\{\text{AQFTs w/ time-slice}\} \simeq \{\mathcal{P}_{\overline{\mathbf{Loc}}[\mathrm{Cauchy}^{-1}]}\text{-algebras in } \mathbf{Alg}_{\mathbb{K}}\}$$

$$\simeq \{\mathcal{P}_{\overline{\mathbf{Loc}}[\mathrm{Cauchy}^{-1}]} \otimes_{\mathrm{BV}} \mathsf{As}\text{-algebras in } \mathbf{Vec}_{\mathbb{K}}\}$$

It is instructive to understand how a $\mathcal{P}_{\overline{\mathrm{Loc}}}$-algebra STRUCTURE realizes the Einstein causality PROPERTY:

Given any causally disjoint $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$,

we have an $\mathrm{Alg}_{\mathbb{K}}$-morphism

$$\begin{aligned} \mathfrak{A}(f_1,f_2) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) &\longrightarrow \mathfrak{A}(N) \\ a_1 \otimes a_2 &\longmapsto a_1 \star a_2 \end{aligned}$$

Because that's an $\mathrm{Alg}_{\mathbb{K}}$-morphism, we find

$$\begin{array}{ccc} \mathfrak{A}(f_1,f_2)\big(a_1a_1' \otimes a_2a_2'\big) & \stackrel{!}{=} & \mathfrak{A}(f_1,f_2)\big(a_1\otimes a_2\big) \cdot \mathfrak{A}(f_1,f_2)\big(a_1'\otimes a_2'\big) \\ \| & & \| \\ (a_1\cdot a_1') \star (a_2\cdot a_2') & & (a_1 \star a_2)\cdot(a_1' \star a_2') \end{array} \qquad (*)$$

With this we compute:

$$\begin{aligned} \mathfrak{A}(f_1)(a_1)\cdot \mathfrak{A}(f_2)(a_2) &= (a_1 \star 1)\cdot(1 \star a_2) \stackrel{(*)}{=} a_1 \star a_2 \stackrel{(*)}{=} (1 \star a_2)\cdot(a_1 \star 1) \\ &= \mathfrak{A}(f_2)(a_2)\cdot \mathfrak{A}(f_1)(a_1) \end{aligned}$$

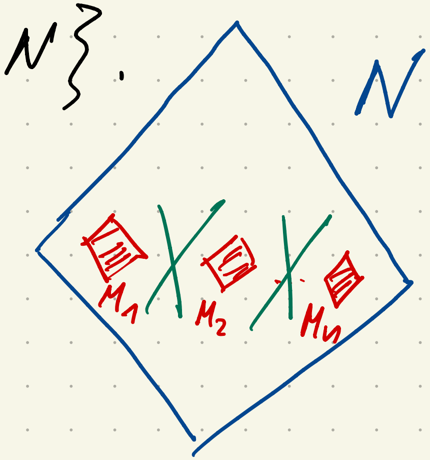
$\rightsquigarrow$ Einstein causality follows from a $\mathcal{P}_{\overline{\mathrm{Loc}}}$-algebra STRUCTURE via an Eckmann–Hilton argument !

Physical interpretation

An AQFT comes with two types of "multiplications" of quantum observables:

1.) "Operator" products $\mu_M : \mathfrak{A}(M) \otimes \mathfrak{A}(M) \longrightarrow \mathfrak{A}(M)$ for each spacetime M

2.) "Factorization" products $\mathfrak{A}(f_1 \dots f_n) : \mathfrak{A}(M_1) \otimes \dots \otimes \mathfrak{A}(M_n) \longrightarrow \mathfrak{A}(N)$
for each family of mutually causally disjoint $\{ f_i : M_i \longrightarrow N \}$.



[Using Fredenhagen's terminology, 2.) could be called a multiplication operation acting on <u>subsystems</u>.]

<u>Importantly</u>: These two types of products are compatible in the sense that $\mathfrak{A}(f_1 \cdots f_n)$ are $\mathbf{Alg}_{\mathbb{K}}$-morphisms.

How is this useful for universal constructions in AQFT?

The following result from operad theory is the key:

Thm: Let $(\mathbf{T},\otimes,I)$ be a bicomplete symmetric monoidal category.

a) For every colored operad $\mathcal{O}$, the category $\mathrm{Alg}_{\mathcal{O}}(\mathbf{T})$ of $\mathcal{O}$-algebras with values in $\mathbf{T}$ is bicomplete.

b) For every morphism $F:\mathcal{O}\longrightarrow\mathcal{P}$ of colored operads, the pullback functor $F^*:\mathrm{Alg}_{\mathcal{P}}(\mathbf{T})\longrightarrow\mathrm{Alg}_{\mathcal{O}}(\mathbf{T})$ admits a left adjoint (operadic left Kan extension).

Denoting by $\mathcal{O}_{\overline{\mathbf{C}}} := \mathcal{P}_{\mathbf{C}} \underset{BV}{\otimes} \mathcal{A}s$ the AQFT operad associated w/ $\overline{\mathbf{C}}=(\mathbf{C},\perp)$ and by $\mathrm{AQFT}(\overline{\mathbf{C}}) := \mathrm{Alg}_{\mathcal{O}_{\overline{\mathbf{C}}}}(\mathrm{Vec}_{\mathbb{K}})$ its category of $\mathrm{Vec}_{\mathbb{K}}$-valued algebras, we get

Cor: a) $\mathrm{AQFT}(\overline{\mathbf{C}})$ is bicomplete, for every orthogonal category $\overline{\mathbf{C}}$.

b) For every orthogonal functor $F:\overline{\mathbf{C}}\longrightarrow\overline{\mathbf{D}}$, there is an adjunction

$$\begin{tikzcd} \mathrm{AQFT}(\overline{\mathbf{C}}) \arrow[r, shift left=1.5, "F_!"] \arrow[r, phantom, "\perp"] & \mathrm{AQFT}(\overline{\mathbf{D}}) \arrow[l, shift left=1.5, "F^*"] \end{tikzcd}$$

Application: Descent in AQFT

Imagine that we have only managed to define an AQFT on the full subcategory $\mathbf{Loc}_\diamond \subseteq \mathbf{Loc}$ of spacetimes whose underlying mnf is diffeomorphic to $\mathbb{R}^m$.

Restricting $\perp$ from $\mathbf{Loc}$ to $\mathbf{Loc}_\diamond$, we get an orthogonal functor $j : \overline{\mathbf{Loc}_\diamond} \longrightarrow \overline{\mathbf{Loc}}$ and hence an adjunction

$$\begin{tikzcd} \mathbf{AQFT}(\overline{\mathbf{Loc}_\diamond}) \arrow[r, shift left=1.5ex, "j_! = \text{extension functor}"] \arrow[r, phantom, "\perp"] & \mathbf{AQFT}(\overline{\mathbf{Loc}}) \arrow[l, shift left=1.5ex, "j^* = \text{restriction functor}"] \end{tikzcd}$$

<u>Proposition:</u> $j_! \dashv j^*$ exhibits $\mathbf{AQFT}(\overline{\mathbf{Loc}_\diamond})$ as a full coreflective subcategory of $\mathbf{AQFT}(\overline{\mathbf{Loc}})$, i.e. the unit $\eta : \mathrm{id} \stackrel{\cong}{\Longrightarrow} j^* j_!$ is a natural isomorphism.

<u>Def:</u> An AQFT $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{Loc}})$ satisfies <u>descent</u> (w.r.t. $\overline{\mathbf{Loc}_\diamond} \xrightarrow{j} \overline{\mathbf{Loc}}$) if the corresponding component of the counit $\varepsilon_{\mathfrak{A}} : j_! j^* \mathfrak{A} \xrightarrow{\cong} \mathfrak{A}$ is an isomorphism.

<u>Example:</u> The free scalar field satisfies this descent condition, but gauge theories do NOT.

In the next lecture, I will introduce a higher categorical generalization of the descent condition that applies to gauge theories.

Summary:

- AQFTs admit an elegant description as algebras over colored operads $\mathcal{O}_{\bar{\mathbf{C}}}$.
- This reformulation turns the Einstein causality and time-slice axioms from PROPERTIES into STRUCTURE, which has various advantages:

  1.) Conceptual interpretation of Einstein causality via Eckmann-Hilton argument.

  2.) Universal constructions preserving the AQFT axioms.

  3.) Starting point for higher categorical AQFT (next lecture!)

- A particularly interesting universal construction is the ext ⊣ res adjunction

$$j_! : \mathbf{AQFT}(\overline{\mathbf{Loc}_\diamond}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{Loc}}) : j^*$$

This is a refinement of Fredenhagen's universal algebra and structurally similar to factorization homology for topological QFTs.

End