Higher structures in algebraic quantum field theory

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Based on joint works w/ M. Benini, S. Bruinsma, M. Perin and L. Woike.

See also arXiv: 1903.02878 for a review.

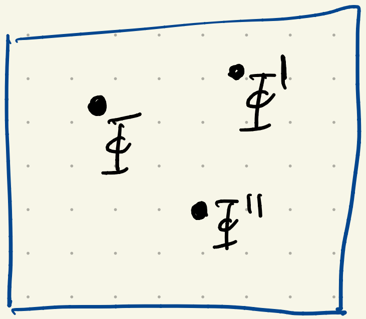
Plan:

- Lecture I: Operads and universal constructions in AQFT
- Lecture II: Higher categories and quantum gauge theories
- Lecture III: Construction of simple examples by homological techniques

II. Higher categories and quantum gauge theories

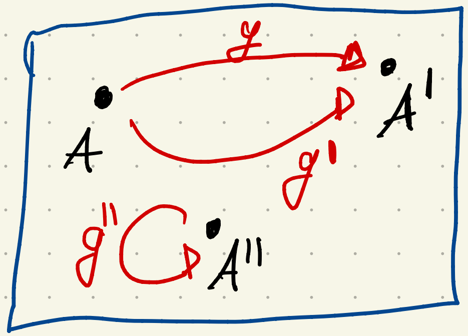
What's the difference between "ordinary" field theories and gauge theories?
It's the way how we compare field configurations!

"Ordinary" field theory



Set/Space of fields, i.e.
being "the same" is a property
$$\Phi = \Phi'$$

Gauge theory



Groupoid/Stack of fields, i.e.
being "the same" is additional structure
$$A \xrightarrow[\cong]{g} A' \quad \text{(gauge transformation)}$$

Important: Moving from "ordinary" field theories to gauge and higher gauge theories increases the dimension of the required category theory:

$$\underset{\text{1-category}}{\mathrm{Set}} \hookrightarrow \underset{\text{2-category}}{\mathrm{Grpd}} \hookrightarrow \underset{\text{3-category}}{2\mathrm{Grpd}} \quad \cdots\cdots \quad \hookrightarrow \underset{\infty\text{-category}}{\infty\mathrm{Grpd}}$$

Model categories: (A hands-on approach to higher category theory)

The crucial feature of higher category theory is that it comes with a weaker concept of equivalence than that of isomorphism in 1-category theory.

Examples:
- weak homotopy equivalences between topological spaces or simplicial sets
- categorical equivalences between groupoids
- quasi-isomorphisms between chain complexes

Model category theory provides a useful framework to deal with such equivalences.

Def: A model category is bicomplete category $\mathcal{C}$ w/ 3 distinguished classes of morphisms:

i) weak equivalences $f: c \xrightarrow{\sim} c'$] that's the main player!

ii) fibrations $f: c \twoheadrightarrow c'$

iii) cofibrations $f: c \hookrightarrow c'$

] needed for working with model categories (e.g. derived functors)

satisfying a long list of axioms.

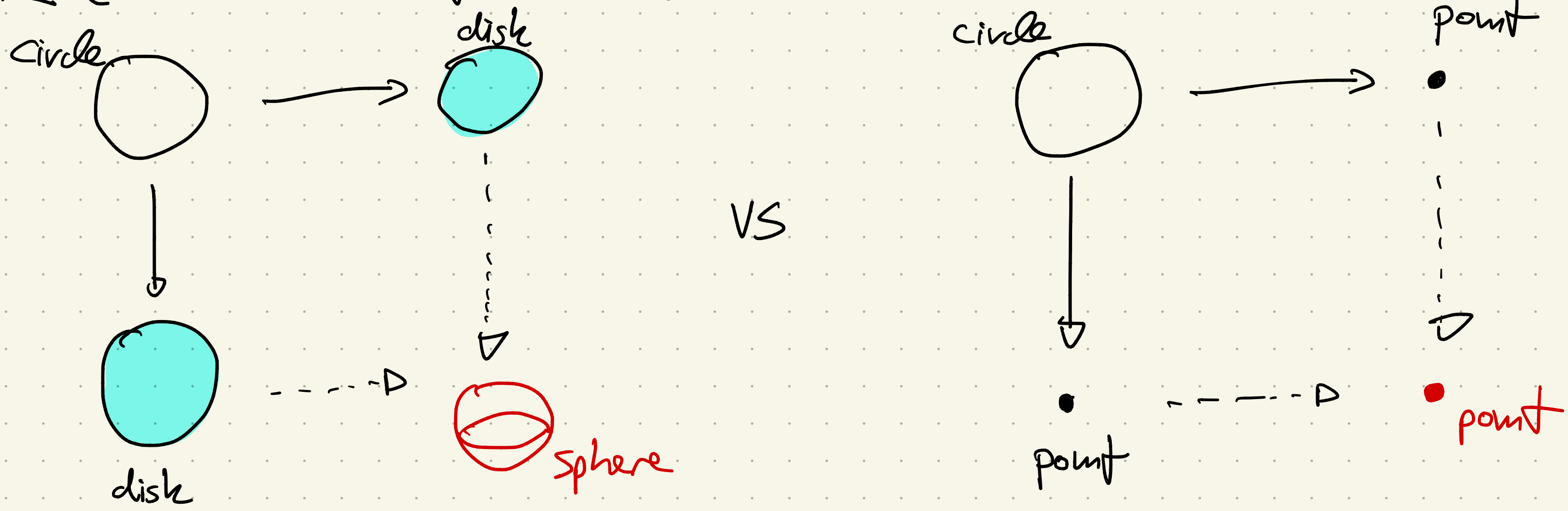
[See Dwyer/Spalinski for an excellent introduction.]

Examples: Top, sSet, Grpd, $Ch_{\mathbb{k}}$ are all model categories with the desired weak equivalences.

Working with model categories is relatively easy (up to the point you want to compute something), but there is one aspect one has to be careful about:

$\boxed{\text{Functors } F: \mathcal{C} \longrightarrow \mathcal{D} \text{ between model categories i.g. do } \underline{\text{NOT}} \text{ preserve weak equivalences!}}$

Example: (Pushout of topological spaces)



This can be solved for a large class of functors by introducing so-called derived functors!

Def: A Quillen adjunction is an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

between model categories, such that G preserves fibrations and acyclic fibrations (or equivalently F preserves cofibrations and acyclic cofibrations).

Thm: For every Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, the left derived functor

$$\mathbb{L}F := F \circ \underbrace{Q}_{\text{cofibrant replacement}} : \mathcal{C} \longrightarrow \mathcal{D}$$

and the right derived functor

$$\mathbb{R}G := G \circ \underbrace{R}_{\text{fibrant replacement}} : \mathcal{D} \longrightarrow \mathcal{C}$$

preserve weak equivalences.

Rem: The main challenge when working with model categories is to find models for derived functors that one can compute. △

Does one really need this mumbo jumbo for studying gauge theories?

- On a Cartesian space $U \cong \mathbb{R}^m$, the groupoid of gauge fields w/ group G is

$$BG_{con}(U) := \begin{cases} \underline{\mathrm{Obj}}: & A \in \Omega^1(U,\mathfrak{g}) \\ \underline{\mathrm{Mor}}: & A \xrightarrow{g} A^g := g^{-1}Ag + g^{-1}dg \,, \text{ with } g \in C^\infty(U,G) \end{cases}$$

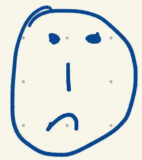
- Given a manifold M w/ good open cover $\{U_i \subseteq M\}$, consider

$$BG_{con}(U_\bullet) := \left(\prod_i BG_{con}(U_i) \rightrightarrows \prod_{ij} BG_{con}(U_{ij}) \mathrel{\substack{\to \\ \to \\ \to}} \prod_{ijk} BG_{con}(U_{ijk}) \mathrel{\substack{\to \\ \to \\ \to \\ \to}} \cdots \right)$$

- Gluing these data with an ordinary limit gives

$$\lim BG_{con}(U_\bullet) \cong \begin{cases} \underline{\mathrm{Obj}}: & A \in \Omega^1(M,\mathfrak{g}) \\ \underline{\mathrm{Mor}}: & A \xrightarrow{g} A^g \,, \text{ with } g \in C^\infty(M,G) \end{cases}$$

no non-trivial bundles?



- Gluing with a homotopy limit $\mathrm{holim} := \mathbb{R}\lim$ (derived functor!) gives

$$\mathrm{holim}\, BG_{con}(U_\bullet) \simeq \begin{cases} \underline{\mathrm{Obj}}: & \left(\{A_i \in \Omega^1(U_i,\mathfrak{g})\}, \{g_{ij} \in C^\infty(U_{ij},G)\}\right) \text{ satisfying} \\ & A_j|_{U_{ij}} = A_i|_{U_{ij}}^{g_{ij}} \quad \text{and} \quad g_{ij}|_{U_{ijk}}\, g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}} \\ \underline{\mathrm{Mor}}: & \{h_i \in C^\infty(U_i,G)\} : \left(\{A_i\}, \{g_{ij}\}\right) \longrightarrow \left(\{A_i^{h_i}\}, \{h_i^{-1}|_{U_{ij}}\, g_{ij}\, h_j|_{U_{ij}}\}\right) \end{cases}$$

all bundles!

Stacks and their function dg-algebras

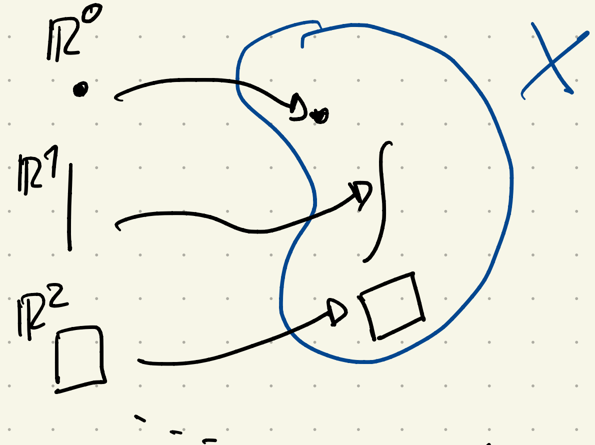
Stacks are suitable generalized spaces to describe a smooth structure on the groupoid of gauge fields.

Without going into the details, a stack is a presheaf $X : \mathrm{Man}^{op} \longrightarrow \mathrm{Grpd}$ satisfying a suitable descent condition (involving holimits).

This is interpreted as a functor of points:

- $X(\mathbb{R}^0)$ = groupoid of points of X
- $X(\mathbb{R}^1)$ = groupoid of smooth curves in X
- $X(\mathbb{R}^2)$ = groupoid of smooth planes in X

...



Example: The stack of gauge fields w/ group G on a Cartesian space $U \cong \mathbb{R}^m$ is

$$\mathrm{Con}_G(U) : \mathrm{Man}^{op} \longrightarrow \mathrm{Grpd}$$

$$N \longmapsto \mathrm{Con}_G(U)(N) = \begin{cases} \underline{\text{Obj}}: & A \in \Omega^{1,0}(U \times N, \mathfrak{g}) \\ \underline{\text{Mor}}: & A \xrightarrow{g} g^{-1}Ag + g^{-1}d_U g \\ & \text{with } g \in C^\infty(U \times N, G) \end{cases}$$

smoothly N-parametrized gauge fields and transformations on U.

Rem: Global points $\mathrm{Con}_G(U)(\mathbb{R}^0) = \mathbf{B}G_{\mathrm{con}}(U)$ give the groupoid from before. Δ

Recall that to every simplicial set one can assign its normalized cochain E_∞-algebra

$$N^*(-,\mathbb{K}) : \mathrm{sSet} \longrightarrow \mathrm{Alg}_{E_\infty}(\mathrm{Ch}_{\mathbb{K}})^{op}.$$

Prop: The above construction generalizes to (∞-)stacks, defining a left Quillen functor

$$N^{\infty *}(-,\mathbb{K}) : \infty\mathrm{Stacks} \longrightarrow \mathrm{Alg}_{E_\infty}(\mathrm{Ch}_{\mathbb{K}})^{op}.$$ (function dg-algebras on stacks)

Examples: – For X a manifold, $N^{\infty *}(X,\mathbb{K}) \simeq C^\infty(X)$ is usual function algebra.

– For $X = Y/\!/\mathfrak{g}$ a Lie algebroid, $N^{\infty *}(X,\mathbb{K}) \simeq CE^\bullet(\mathfrak{g}, C^\infty(Y))$ is Chevalley–Eilenberg cochain algebra.

– For $X = Y/\!/G$ a Lie groupoid, $N^{\infty *}(X,\mathbb{K}) \simeq C^\bullet(G, C^\infty(Y))$ is group cochain algebra.

Warning: In general, $N^{\infty *}(X,\mathbb{K})$ does NOT encode the stack X faithfully.

For example, $N^{\infty *}(*/\!/\mathbb{Z}_2, \mathbb{K}) \simeq \mathbb{K} \simeq N^{\infty *}(*,\mathbb{K})$.

This problem is due to finite gauge transformations. (Infinitesimal ones work fine.)

The solution is to assign instead QCoh-sheaf categories $\mathrm{QCoh} : \infty\mathrm{Stacks} \longrightarrow \mathrm{SMdgCat}^{op}$.

[See Calaque, Pantev, Toën, Vaquié, Vezzosi for details and arXiv:2003.13713 for some steps to incorporate this in AQFT.]

Homotopy AQFTs with values in $\mathrm{Ch}_{\mathbb{K}}$

The previous discussion motivates us to consider AQFTs with values in chain complexes to describe quantum gauge theories in terms of "quantized function algebras" on (co-)stacks. (This might neglect some non-perturbative features, but it is much easier than working with QCoh.)

Due to our operadic approach, this is quite easy:

Def: The category of strict $\mathrm{Ch}_{\mathbb{K}}$-valued AQFTs on an orthogonal category $\overline{\mathcal{C}} = (\mathcal{C}, \perp)$ is the category of $\mathrm{Ch}_{\mathbb{K}}$-valued algebras over the AQFT operad $\mathcal{O}_{\overline{\mathcal{C}}}$

$$\mathrm{AQFT}_{\mathrm{strict}}(\overline{\mathcal{C}}) := \mathrm{Alg}_{\mathcal{O}_{\overline{\mathcal{C}}}}(\mathrm{Ch}_{\mathbb{K}}) \quad .$$

Examples: Models constructed via the perturbative BV-formalism [Fredenhagen/Rejzner] define strict $\mathrm{Ch}_{\mathbb{K}}$-valued AQFTs on $\mathbf{Loc}$.

Thm: $\mathrm{AQFT}_{\mathrm{strict}}(\overline{\mathcal{C}})$ is a model category with the following choices:

i) $\xi : A \Rightarrow B$ is weak equivalence if all $\xi_c : A(c) \longrightarrow B(c)$ are quasi-isos

ii) $\xi : A \Rightarrow B$ is fibration if all $\xi_c : A(c) \longrightarrow B(c)$ are degree-wise surjective.

iii) cofibrations are determined by the left lifting property.

Homotopy-coherent AQFTs

associative algebra	A_∞-algebra
Lie algebra	L_∞-algebra
AQFT	???

It is useful to develop a concept of homotopy AQFTs, because many constructions will not (directly) give strict AQFTs. (Recall e.g. that stacks have function E_∞-algebras!)

Again, operad theory tells us how to do this:

Def: i) A colored dg-operad $\mathcal{O}$ is called Σ-cofibrant if each component $\mathcal{O}\binom{t}{\underline{c}}$ is a cofibrant object in $\mathrm{Fun}(\Sigma_{\underline{c}}, \mathrm{Ch}_{\mathbb{K}})$, with $\Sigma_{\underline{c}}$ the stabilizer group of $\underline{c} = (c_1 \cdots c_n)$.

ii) A Σ-cofibrant resolution of a colored dg-operad $\mathcal{O}$ is a Σ-cofibrant dg-operad $\mathcal{O}_\infty$ with an acyclic fibration $\mathcal{O}_\infty \xrightarrow{\sim} \mathcal{O}$.

iii) The model category of homotopy $\mathcal{O}$-algebras is defined as

$$\mathrm{Alg}_{\mathcal{O}_\infty}(\mathrm{Ch}_{\mathbb{K}}) \;, \text{ with } \mathcal{O}_\infty \xrightarrow{\sim} \mathcal{O} \text{ any } \Sigma\text{-cofibrant resolution.}$$

Rem: Up to Quillen equivalence, $\mathrm{Alg}_{\mathcal{O}_\infty}(\mathrm{Ch}_{\mathbb{K}})$ does not depend on the choice of Σ-cofibrant resolution. $\triangle$

<u>Def</u>: The model category of homotopy AQFTs on $\overline{\mathbf{C}}$ is defined by

$$\mathbf{AQFT}_\infty(\overline{\mathbf{C}}) := \mathbf{Alg}_{\mathcal{O}_{\overline{\mathbf{C}}\infty}}(\mathbf{Ch}_{\mathbb{K}}) ,$$

where $\mathcal{O}_{\overline{\mathbf{C}}\infty} \xrightarrow{\sim} \mathcal{O}_{\overline{\mathbf{C}}}$ is a Σ-cofibrant resolution of the AQFT operad.

<u>Example</u>: The component-wise tensor product $\mathcal{O}_{\overline{\mathbf{C}}} \otimes E_\infty \xrightarrow{\sim} \mathcal{O}_{\overline{\mathbf{C}}}$ with the Barratt-Eccles E_∞-operad defines a Σ-cofibrant resolution.

This type of homotopy AQFTs is useful to capture E_∞-algebras on (∞-)stacks.

<u>Strictification Theorem</u>: (For $\mathrm{char}\,\mathbb{K} = 0$, as assumed throughout this lecture series)

The identity $\mathrm{id}: \mathcal{O}_{\overline{\mathbf{C}}} \xrightarrow{\sim} \mathcal{O}_{\overline{\mathbf{C}}}$ is a Σ-cofibrant resolution, hence

$\mathbf{AQFT}_\infty(\overline{\mathbf{C}}) \simeq \mathbf{AQFT}_{\mathrm{strict}}(\overline{\mathbf{C}})$ are Quillen equivalent.

<u>In words</u>: $\boxed{\text{Every homotopy AQFT is weakly equivalent to a strict one.}}$

Application: Higher categorical descent in homotopy AQFT

Prop: Every orthogonal functor $F: \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ defines a Quillen adjunction

$$F_! : \mathbf{AQFT}_\infty(\overline{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}_\infty(\overline{\mathbf{D}}) : F^*$$

Example: For $j: \overline{\mathbf{Loc}_\Diamond} \to \overline{\mathbf{Loc}}$, we get a derived extension functor

$$\mathbb{L}j_! : \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}_\Diamond}) \longrightarrow \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}})$$

that can be used to extend quantum gauge theories from "diamonds" $M \cong \mathbb{R}^m$ to all spacetimes.

Toy-model: Let $\overline{\mathbf{C}} = (\mathbf{Disk}_2, \perp_{max}) \xrightarrow{j} \overline{\mathbf{D}} = (\mathbf{Man}_2, \perp_{max})$, then

$$\mathbb{L}j_! : \mathbf{Fun}(\mathbf{Disk}_2, \mathbf{Alg}_{E_\infty}(\mathbf{Ch}_\mathbb{K})) \longrightarrow \mathbf{Fun}(\mathbf{Man}_2, \mathbf{Alg}_{E_\infty}(\mathbf{Ch}_\mathbb{K})).$$

[Extension from oriented 2-disks to oriented 2-mfds.]

Consider
$$\mathfrak{A}: \mathbf{Disk}_2 \longrightarrow \mathbf{Alg}_{E_\infty}(\mathbf{Ch}_\mathbb{K}), \quad U \longmapsto \mathbb{E}_\infty(\mathbb{K}[-1])$$
which describes a theory with only a constant ghost field

Then one can compute on $M \in \mathbf{Man}_2$:

$$(\mathbb{L}j_!\mathfrak{A})(M) \simeq \mathbb{E}_\infty\big(\Omega_c^\bullet(M)[1]\big)$$

(This is unquantized linear Chern-Simons on the oriented surface $M \in \mathbf{Man}_2$!)

Summary:

- Higher categorical structures in classical gauge theory arise due to the groupoid/stack structure of gauge fields and their gauge transformations.
- Model category theory is a (relatively) concrete framework to capture and work with such higher structures.
- In quantum gauge theory, the higher categorical structures are encoded in dg-algebras of quantum observables, as e.g. those appearing in the BV-formalism
- We developed model categories for strict $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs and for homotopy AQFTs, together with derived universal constructions.
- Derived local-to-global extensions are interesting and rich!
  For example, the unquantized linear Chern-Simons algebra $\mathbb{E}_\infty(\Omega^\bullet_c(M)[1])$ on an oriented surface $M \in \mathbf{Man}_2$ can be obtained by extending a very simple theory on $\mathbf{Disk}_2$ that describes a constant ghost field.

End