Higher structures in algebraic quantum field theory

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Based on joint works w/ M. Benini, S. Bruinsma, M. Perin and L. Woike.

See also arXiv: 1903.02878 for a review.

Plan:

- Lecture I: Operads and universal constructions in AQFT
- Lecture II: Higher categories and quantum gauge theories
- Lecture III: Construction of simple examples by homological techniques

III. Construction of simple examples by homological techniques

In this lecture, I explain how to construct a simple class of homotopy AQFTs:

$\boxed{\text{Linear quantum gauge theories}}$

Our construction is inspired by derived algebraic geometry, but it is much simpler because for linear gauge theories we can use chain complexes instead of derived stacks.

Our construction is a mathematical formalization of the BV-formalism, which I believe has the following advantages:

1.) All steps are described by derived functors, hence are compatible with quasi-isomorphisms.

2.) We do NOT need "tricks" like gauge fixing and Nakanishi-Lautrup fields.

3.) We clarify the origin of unshifted Poisson structures in Lorentzian signature.

Unfortunately, we can currently treat only linear gauge theories, while the standard BV-formalism works for perturbative gauge theories.

Conventions for chain complexes

- We work in homological degree conventions, i.e. differentials are of degree -1:

$$V = \left(\cdots \xleftarrow{d} V_{-1} \xleftarrow{d} V_0 \xleftarrow{d} V_1 \xleftarrow{d} \cdots \right) \in \mathrm{Ch}_{\mathbb{K}} .$$

- The $p \in \mathbb{Z}$ shifted complex $V[p]$ is defined by $V[p]_n := V_{n-p}$ and $d^{V[p]} := (-1)^p d^V$.

- Given two chain complexes $V, W \in \mathrm{Ch}_{\mathbb{Z}}$, we have a mapping complex

$$\underline{\hom}(V,W) \in \mathrm{Ch}_{\mathbb{K}} \quad \text{defined by} \quad \underline{\hom}(V,W)_n := \prod_{m \in \mathbb{Z}} \mathrm{Lin}(V_m, W_{n+m})$$

and the differential $\partial \{ L_m : V_m \to W_{n+m} \}_{m \in \mathbb{Z}} := \{ d\, L_m - (-1)^n L_{m-1} d : V_m \to W_{n-1+m} \}_{m \in \mathbb{Z}}$

- Note that:

  i) chain maps $f : V \to W$ are 0-cycles, i.e. $f \in \underline{\hom}(V,W)_0$ s.t. $\partial f = 0$

  ii) chain homotopies between $f, g : V \to W$ are 1-chains $\lambda \in \underline{\hom}(V,W)_1$

  s.t. $f - g = \partial \lambda$

  ... and similar for higher chain homotopies.

Field and solution complex (for illustrative purposes, I will focus on linear YM theory)

Linear Yang-Mills theory on a spacetime $M \in \mathbf{Loc}$ is specified by the following data:

(1) Field complex: $\mathcal{F}(M) = \Big(\underbrace{\overset{(0)}{\Omega^1(M)}}_{\text{gauge fields}} \xleftarrow{\ d\ } \underbrace{\overset{(1)}{\Omega^0(M)}}_{\text{gauge transformations}} \Big)$

(2) Gauge invariant action: $S(A) = \int_M \frac{1}{2}\, dA \wedge * dA$

The variation $\delta^{var} S$ of S defines a section of the cotangent bundle

$$
\begin{tikzcd}
\mathcal{F}(M) \arrow[d, "\delta^{var} S"'] \\
T^*\mathcal{F}(M)
\end{tikzcd}
\quad = \quad
\left(
\begin{tikzcd}
\overset{(-1)}{0} \arrow[d, "0"'] & \overset{(0)}{\Omega^1(M)} \arrow[l, "0"'] \arrow[d, "{(\mathrm{id}, \delta d)}", "\text{linear YM eqn}"'] & \overset{(1)}{\Omega^0(M)} \arrow[l, "d"'] \arrow[d, "\mathrm{id}"] \\
\Omega^0(M) & \Omega^1(M) \times \Omega^1(M) \arrow[l, "-\delta \pi_2"] & \Omega^0(M) \arrow[l, "i_1 d"]
\end{tikzcd}
\right)
$$

Def: The solution complex is the (linear) derived critical locus

$$
\begin{tikzcd}
\mathrm{Sol}(M) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\text{homotopy pullback}"] & \mathcal{F}(M) \arrow[d, "\delta^{var} S"] \\
\mathcal{F}(M) \arrow[r, "0"'] & T^*\mathcal{F}(M)
\end{tikzcd}
$$

Proposition: A model for the solution complex is

$$\mathrm{Sol}(M) = \Big(\underbrace{\overset{(-2)}{\Omega^0(M)}}_{c^+} \xleftarrow{\ \delta\ } \underbrace{\overset{(-1)}{\Omega^1(M)}}_{A^+} \xleftarrow{\ \delta d\ } \underbrace{\overset{(0)}{\Omega^1(M)}}_{A} \xleftarrow{\ d\ } \underbrace{\overset{(1)}{\Omega^0(M)}}_{c} \Big)$$

Rem: $\mathrm{Sol}(M)$ describes the field content and differentials known from the $\boxed{\text{BV-formalism}}$.

Sketch of proof: To compute the homotopy pullback, replace the 0-section

$$0 : \mathfrak{F}(M) \longrightarrow T^*\mathfrak{F}(M) = \mathfrak{F}(M) \times \mathfrak{F}_c(M)^*$$ by a weakly equivalent fibration.

One way to do this is to introduce the acyclic complex

$$D = \big(\overset{(-1)}{\mathbb{R}} \xleftarrow{\ \mathrm{id}\ } \overset{(0)}{\mathbb{R}} \big)$$ and note that $0 \overset{\sim}{\hookrightarrow} D \twoheadrightarrow \mathbb{R}$.

This gives a factorization

$$\begin{tikzcd}
\mathfrak{F}(M) \arrow[rr, "0"] \arrow[d, "\sim"'] & & \mathfrak{F}(M) \times \mathfrak{F}_c(M)^* \\
\mathfrak{F}(M) \times (D \otimes \mathfrak{F}_c(M)^*) \arrow[urr, "\tilde{0}"'] & &
\end{tikzcd}$$

(fibration!)

and we can compute $\mathrm{Sol}(M)$ by the ordinary pullback

$$\begin{tikzcd}
\mathrm{Sol}(M) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{F}(M) \arrow[d, "\delta^{\mathrm{vars}} S"] \\
\mathfrak{F}(M) \times (D \otimes \mathfrak{F}_c(M)^*) \arrow[r, "\tilde{0}"'] & T^*\mathfrak{F}(M)
\end{tikzcd}$$

$\square$

Rem: The homology groups of the solution complex are:

- $H_1(\mathrm{Sol}(M)) \cong H^0_{dR}(M)$, the stabilizer group of gauge fields
- $H_0(\mathrm{Sol}(M)) \cong \dfrac{\{A \in \Omega^1(M) : \delta d A = 0\}}{d\Omega^0(M)}$, the gauge orbit space of solutions
- $H_{-1}(\mathrm{Sol}(M)) \cong H^{m-1}_{dR}(M)$, the obstructions to solve $\delta d A = j$
- $H_{-2}(\mathrm{Sol}(M)) \cong H^m_{dR}(M) = 0$, because $M \cong \mathbb{R} \times \Sigma$ $\triangle$

In particular:

The solution complex $\mathrm{Sol}(M)$ contains more refined information than the gauge orbit space of solutions $H_0(\mathrm{Sol}(M))$.

These are both stacky and derived higher structures.

positive degrees

negative degrees

Shifted and unshifted Poisson structures

Every derived critical locus has a $[-1]$-shifted Poisson structure (called antibracket in BV-formalism)

[See CPTVV for general results in derived algebraic geometry and Costello/Gwilliam in the context of QFT.]

For linear gauge theories, this is simply a chain map

$$\gamma : \mathcal{L}(M) \otimes \mathcal{L}(M) \longrightarrow \mathbb{R}[1]$$

on the dual complex $\mathcal{L}(M)$ of linear observables on $\mathrm{Sol}(M)$.

Concretely for our example:

- $$\mathcal{L}(M) = \Big(\overset{(-1)}{\Omega^0_c(M)} \xleftarrow{-\delta} \overset{(0)}{\Omega^1_c(M)} \xleftarrow{\delta d} \overset{(1)}{\Omega^1_c(M)} \xleftarrow{-d} \overset{(2)}{\Omega^0_c(M)} \Big)$$

- $$\begin{tikzcd} \mathcal{L}(M) \otimes \mathcal{L}(M) \arrow[r, "\gamma"] \arrow[d, "\mathrm{id} \otimes i"'] & \mathbb{R}[1] \\ \mathcal{L}(M) \otimes \mathrm{Sol}(M)[1] \arrow[r, "\cong"'] & \big(\mathcal{L}(M) \otimes \mathrm{Sol}(M)\big)[1] \arrow[u, "\langle \cdot , \cdot \rangle"'] \end{tikzcd}$$

canonical inclusion of $\mathcal{L}(M)$ into $\mathrm{Sol}(M)[1]$

duality pairing

In our setting where $M \in \mathbf{Loc}$ is a globally hyperbolic Lorentzian mnf something funny happens:

<u>Lem</u>: The canonical inclusion

$$\begin{tikzcd} \mathfrak{L}(M) \arrow[d, "i"] \\ \mathrm{Sol}(M)[1] \end{tikzcd} = \left(\begin{tikzcd} \Omega^0_c(M) \arrow[d, "\subseteq"] & \Omega^1_c(M) \arrow[l, "-\delta"'] \arrow[d, "\subseteq"] & \Omega^1_c(M) \arrow[l, "\delta d"'] \arrow[d, "-\subseteq"] & \Omega^0_c(M) \arrow[l, "-d"'] \arrow[d, "-\subseteq"] \\ \Omega^0(M) & \Omega^1(M) \arrow[l, "-\delta"] & \Omega^1(M) \arrow[l, "-\delta d"] & \Omega^0(M) \arrow[l, "-d"] \end{tikzcd} \right)$$

defines a trivial homology class $[i] = 0 \in H_0\big(\underline{\mathrm{hom}}\,(\mathfrak{L}(M), \mathrm{Sol}(M)[1])\big)$.

Hence, also the shifted Poisson structure defines a trivial homology class

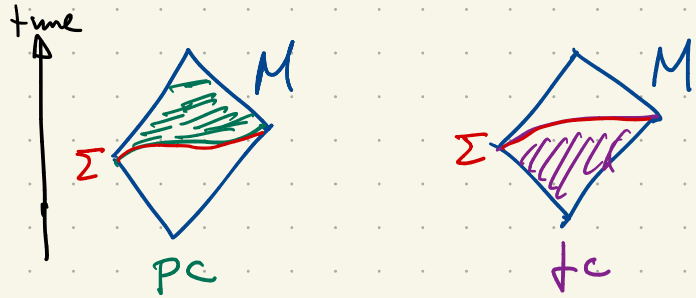
$$[\Upsilon] = 0 \in H_0\Big(\underline{\mathrm{hom}}\,\big(\mathfrak{L}(M) \otimes \mathfrak{L}(M), \mathbb{R}[1]\big)\Big).$$

<u>Rem</u>: Proving this result uses heavily the solution theory of the Maxwell operator δd on globally hyperbolic Lorentzian manifolds M. $\triangle$

So does this mean that the shifted Poisson structure (antibracket) is useless, i.e. without any actual content?

Not quite! There exist two distinct, intrinsically Lorentzian, ways to trivialize i, which make use of past/future compact support systems:

$$\begin{tikzcd} \mathcal{L}(M) \arrow[rr, "i"] \arrow[dr, "\subseteq"'] & & \mathrm{Sol}(M)[1] \\ & \mathcal{L}_{pc/fc}(M) \arrow[ur, "i_{pc/fc}"'] & \end{tikzcd}$$



Thm: a) There exists a unique (up to contractible choices) contracting homotopy $\mathcal{G}^{\pm}$ for $\mathcal{L}_{pc/fc}(M)$, e.g.

$$\begin{tikzcd} 0 \arrow[d, "\mathrm{id}"] \arrow[dr, "0"] & \Omega^0_{pc/fc}(M) \arrow[l, "0"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "-G^{\pm}_{\square} d"] & \Omega^1_{pc/fc}(M) \arrow[l, "-\delta"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "G^{\pm}_{\square}"] & \Omega^1_{pc/fc}(M) \arrow[l, "\delta d"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "-\delta G^{\pm}_{\square}"] & \Omega^0_{pc/fc}(M) \arrow[l, "-d"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "0"] & 0 \arrow[l, "0"'] \arrow[d, "\mathrm{id}"] \\ 0 & \Omega^0_{pc/fc}(M) \arrow[l, "0"] & \Omega^1_{pc/fc}(M) \arrow[l, "-\delta"] & \Omega^1_{pc/fc}(M) \arrow[l, "\delta d"] & \Omega^0_{pc/fc}(M) \arrow[l, "-d"] & 0 \arrow[l, "0"] \end{tikzcd}$$

where $G^{\pm}_{\square}$ is the retarded/advanced Green's operator for the d'Alembertian $\square = \delta d + d\delta$.

b) The difference $\mathcal{G} := i_{pc} \circ \mathcal{G}^{+} - i_{fc} \circ \mathcal{G}^{-} : \mathcal{L}(M) \longrightarrow \mathrm{Sol}(M)$ defines a non-trivial homology class $[\mathcal{G}] \in H_0(\underline{\mathrm{hom}}(\mathcal{L}(M), \mathrm{Sol}(M)))$.

As a consequence, we can define a non-trivial unshifted Poisson structure

$$\tau : \mathcal{L}(M) \otimes \mathcal{L}(M) \xrightarrow{\mathrm{id} \otimes \mathcal{G}} \mathcal{L}(M) \otimes \mathrm{Sol}(M) \xrightarrow{\langle\, ,\, \rangle} \mathbb{R}$$

which is unique up to contractible choices.

Thm: The canonical quantization functor

$$\begin{aligned} \mathrm{CCR} : \mathrm{PoCh}_{\mathbb{R}} &\longrightarrow \mathrm{dgAlg}_{\mathbb{C}} \\ (V, \tau) &\longmapsto T^{\otimes}_{\mathbb{C}} V \Big/ \left\langle v \otimes v' - (-1)^{|v||v'|} v' \otimes v - i\hbar\, \tau(v, v') \right\rangle \end{aligned}$$

preserves weak equivalences, i.e. it does NOT have to be derived.

Furthermore, if τ and $\tau + \partial \rho$ are homotopic Poisson structures, then there exists a zig-zag of weak equivalences in $\mathrm{dgAlg}_{\mathbb{C}}$

$$\mathrm{CCR}(V, \tau) \xleftarrow{\sim} A_{(V, \tau, \rho)} \xrightarrow{\sim} \mathrm{CCR}(V, \tau + \partial \rho)$$

In words: $\boxed{\text{Quantization is consistent in our homological approach.}}$

Main Theorem:

The assignment $M \longmapsto \mathfrak{A}(M) := CCR(\mathcal{L}(M), \tau)$ defines a semi-strict homotopy AQFT $\mathfrak{A} \in \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}})$.

In more detail, the functor $\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathbf{dgAlg}_{\mathbb{C}}$ satisfies

i) <u>Strict Einstein causality</u>: For every $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$, the chain map

$$\underbrace{[\cdot\,,\cdot]_{\mathfrak{A}(N)}}_{\text{graded commutator}} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N)$$

is zero.

ii) <u>Homotopy time-slice axiom</u>: For every Cauchy morphism $f : M \to N$, the $\mathbf{dgAlg}_{\mathbb{C}}$-morphism $\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\sim} \mathfrak{A}(N)$ is a weak equivalence.

Summary:

- Examples of homotopy AQFTs can be constructed by adapting techniques from derived algebraic geometry to applications in field theory.
- At the moment we can do this only for linear (higher) gauge theories, because there the problem reduces to basic homological algebra.
- Even though our approach looks superficially similar to the BV-formalism, I believe that it has some advantages:
  1.) All constructions are derived, i.e. compatible with weak equivalences.
  2.) No "tricks" like gauge fixing and Nakanishi-Lautrup fields needed.
  3.) Relationship between trivializations of the shifted Poisson structure and the unshifted Poisson structure entering CCR-quantization.
- Open problem: Generalize these constructions to perturbative quantum gauge theories w/ interactions.

End