Derived Algebraic Geometry in Mathematical Physics

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Joint with Benini/Safronov [2104.14886] and Benini/Pridham [2201.10225].

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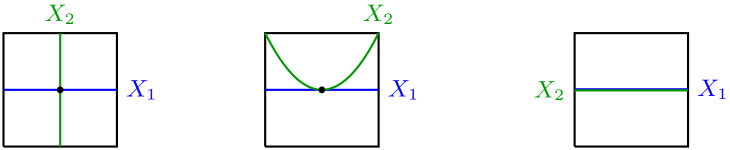
(ii) Non-transversal intersections:

$X_1 \times_Y X_2$ is in general singular. It also ignores intersection multiplicities (in the case of manifolds) and may violate the codimension addition rule, e.g.



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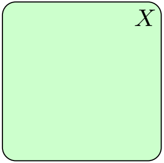

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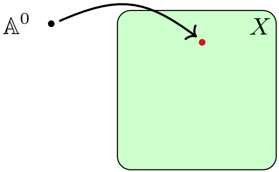

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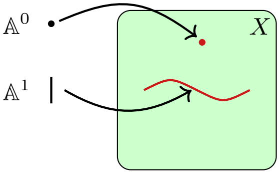

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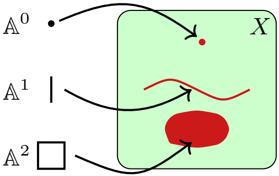

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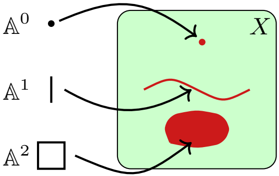
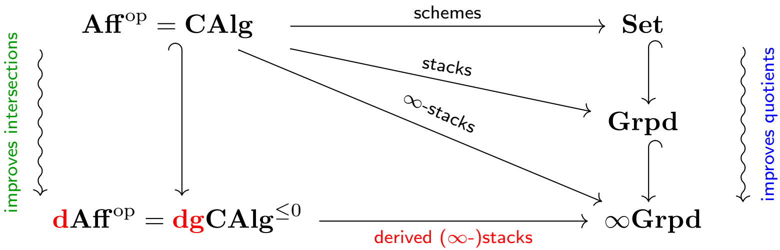


◇ Derived stacks have a richer functor of points:



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◇ Many problems in mathematical physics require intersections and quotients!

A typical problem is to describe the space of solutions of a variational PDE ($\hat{=}$ intersection $\mathrm{d}^{\mathrm{dR}} S = 0$), modulo gauge symmetries ($\hat{=}$ quotient).

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◇ Predecessors of DAG were successfully used in physics since the 80s!

It is now understood that the BRST/BV/BFV/... formalisms from physics capture formal (perturbative) aspects of derived geometry:

$$\left\{ \begin{array}{c} \text{formal neighborhood of a point} \\ x : \mathrm{pt} \to X \text{ in a derived stack } X \end{array} \right\} \overset{\text{Lurie}}{\underset{\text{Pridham}}{\Longleftrightarrow}} L_\infty\text{-algebra}$$

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! Interesting for modern developments in QFT, e.g. in the context of factorization algebras [Costello/Gwilliam; ...] or homotopical algebraic QFT [Benini/AS; ...].

Application 1:

Derived critical locus of a function on a quotient stack

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◇ **Wanted:** Explicit model for the space of critical points of S, i.e. the derived critical locus

$$\begin{array}{ccc} \mathrm{dCrit}(S) & \dashrightarrow & [X/G] \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ [X/G] & \xrightarrow[\mathrm{d}^{\mathrm{dR}} S]{} & T^*[X/G] \end{array}$$

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Rem: The case where $G = \mathrm{pt}$ is trivial was worked out by [Vezzosi] and formal quotient stacks $[X/\mathfrak{g}]$ for Lie algebras were studied by [Costello/Gwilliam].

Theorem [Benini, Safronov, AS]

The derived critical locus $\mathrm{dCrit}(S : [X/G] \to \mathbb{A}^1) \simeq [Z/G]$ is a derived quotient stack with $Z = \mathsf{Spec}\ \mathcal{O}^\bullet(Z)$ the derived affine scheme specified by the CDGA

$$\mathcal{O}^\bullet(Z) \,=\, \mathrm{Sym}_A\Big(\mathsf{T}_A[1] \oplus \big(A \otimes \mathfrak{g}[2]\big)\Big)$$

$$\partial a \,=\, 0 \quad , \quad \partial v = \iota_v(\mathrm{d}^{\mathrm{dR}} S) \quad , \quad \partial t = -\iota_{\rho(t)}(\lambda) \quad ,$$

for all $a \in A$, derivations $v \in \mathsf{T}_A[1]$ and Lie algebra elements $t \in \mathfrak{g}[2]$. Here $\lambda \in \Omega^1(T^*X)$ denotes the tautological 1-form and ρ is the induced Lie algebra action on T^*X.

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Physics note: This result matches the expectations from the BV formalism. In particular, one clearly recognizes the fields A, the anti-fields $\mathsf{T}_A[1]$ and the anti-ghosts $\mathfrak{g}[2]$. The ghosts G are encoded non-perturbatively by the quotient stack.

Some remarks and observations

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(3) Our result can be compared to the perturbative BV formalism:

$$\begin{array}{c} \mathcal{O}^\bullet\big(\mathrm{dCrit}(S)\big) \simeq \mathcal{O}^\bullet\big([Z/G]\big) \simeq \mathrm{Tot}^\Pi \overbrace{\mathsf{N}^\bullet\big(G, \mathcal{O}^\bullet(Z)\big)}^{\text{normalized group cochains}} \\ \Big\downarrow \text{i.g. } \not\simeq \ \text{van Est map} \\ \mathcal{O}^\bullet\big(\mathrm{BV}(S)\big) \simeq \mathcal{O}^\bullet\big([Z/\mathfrak{g}]\big) \simeq \mathrm{Tot}^\Pi \underbrace{\mathsf{CE}^\bullet\big(\mathfrak{g}, \mathcal{O}^\bullet(Z)\big)}_{\text{Chevalley-Eilenberg cochains}} \end{array}$$

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(4) $\mathrm{dCrit}(S) \simeq [Z/G]$ is i.g. **not** affine, i.e. not determined by $\mathcal{O}^\bullet\big(\mathrm{dCrit}(S)\big)$. Need richer algebraic invariant such as SM dg-category of modules

$$\mathbf{QCoh}\big(\mathrm{dCrit}(S)\big) \simeq {}_{\mathcal{O}^\bullet(Z)}\mathbf{dgMod}^G$$

Application 2:

Quantization of derived cotangent stacks

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- **Wanted:** Quantization of $T^*[X/G]$ along this Poisson structure. Find an explicit model for the quantized E_0-monoidal ($=$ pointed) dg-category

$$\mathbf{QCoh}\big(T^*[X/G]\big)_\hbar = \textbf{?}$$

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$$\mathsf{CE}^\bullet\big(\mathfrak{g}, \mathcal{O}^\bullet(\mu^{-1}(0))\big)_\hbar \rightrightarrows \mathsf{CE}^\bullet\big(\mathfrak{g} \oplus \mathfrak{g}, \mathcal{O}^\bullet(\mu^{-1}(0) \times G)\big)_\hbar \;\overrightarrow{\overrightarrow{\rightarrow}}\; \cdots$$

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(3) Obtain global quantization by computing homotopy limit of dg-categories

$$\mathbf{QCoh}\big(T^*[X/G]\big)_\hbar := \mathrm{holim}\, \circledast \in \mathbf{dgCat}$$

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- *Objects:* Triples $(\mathcal{E}^\bullet, \nabla, \Psi)$ consisting of
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These data have to satisfy the following conditions

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- *Morphisms:* $\underline{\mathrm{hom}}_{\mathcal{O}(X)[[\hbar]]}(\mathcal{E}^\bullet, \mathcal{E}'^\bullet)$ preserving G, ∇ and Ψ strictly

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◇ <u>*Objects:*</u> Triples $(\mathcal{E}^\bullet, \nabla, \Psi)$ consisting of

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- a G-eqv. dg-connection $\nabla : \mathcal{E}^\bullet \to \Omega^1(X)[[\hbar]] \otimes_{\mathcal{O}(X)[[\hbar]]} \mathcal{E}^\bullet$ with respect to $\hbar\, \mathrm{d}^{\mathrm{dR}}$, i.e. $\nabla(a\, s) = \hbar\, \mathrm{d}^{\mathrm{dR}} a \otimes s + a\, \nabla(s)$
- a G-eqv. graded module map $\Psi : \mathfrak{g}[1] \otimes \mathcal{E}^\sharp \to \mathcal{E}^\sharp$

These data have to satisfy the following conditions

$$\nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar \nabla_{[v,v']} \quad , \quad \nabla_v \Psi_t - \Psi_t \nabla_v = 0$$
$$\Psi_t \Psi_{t'} + \Psi_{t'} \Psi_t = 0 \qquad \partial \Psi_t + \Psi_t \partial = \nabla_{\mu^*(t)} + \hbar\, \rho(t)$$

◇ <u>*Morphisms:*</u> $\underline{\hom}_{\mathcal{O}(X)[[\hbar]]}(\mathcal{E}^\bullet, \mathcal{E}'^\bullet)$ preserving G, ∇ and Ψ strictly

Physics note: The dg-modules $\mathcal{E}^\bullet$ play the role of "spaces of wave functions". The G-action encodes the ghosts. The connection ∇ describes the canonical momentum operators and Ψ defines an action of the anti-ghosts. The $\hbar$ corrections have the same pattern as the canonical commutation relations in the formal (perturbative) setting.

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2. Study and work out further examples of deformation quantizations of unshifted and also shifted Poisson structures on derived stacks.

   Interesting candidate: $[\mathrm{pt}/G]$ for a higher group $\rightsquigarrow$ higher quantum groups? [work in progress with Laugwitz].