Quantum field theories on Lorentzian manifolds

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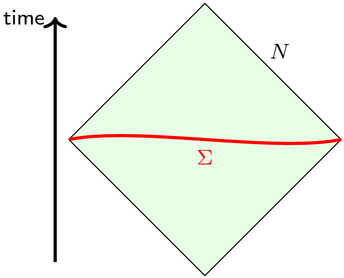
Based on a research program with Marco Benini, with contributions from

S. Bruinsma, S. Bunk, V. Carmona, C. Fewster, L. Giorgetti, A. Grant-Stuart, J. MacManus, G. Musante, M. Perin, J. Pridham, P. Safronov, U. Schreiber, R. Szabo and L. Woike.

Some background on Lorentzian spacetimes

- **Spacetime** $:=$ oriented and time-oriented globally hyperbolic Lorentzian manifold N



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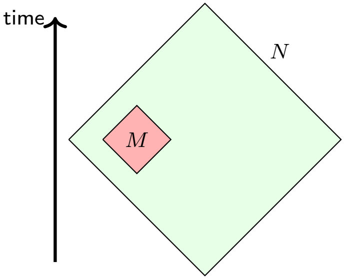

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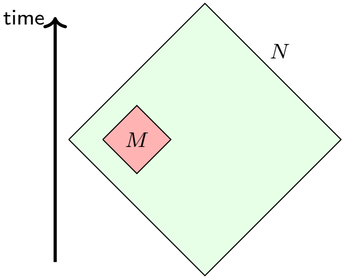

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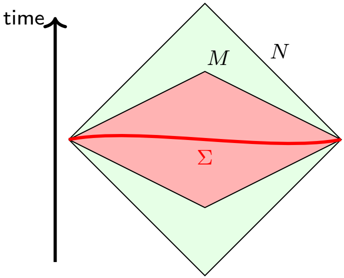

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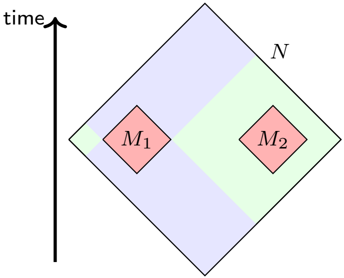

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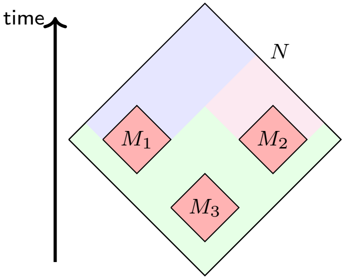

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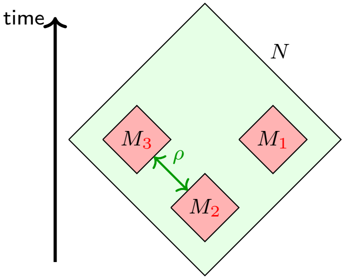

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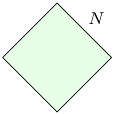

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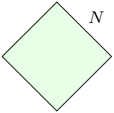

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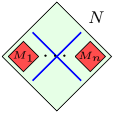

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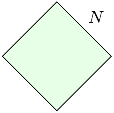

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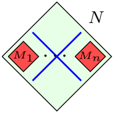

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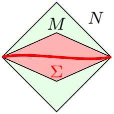

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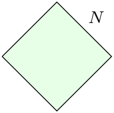

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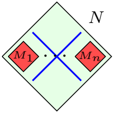

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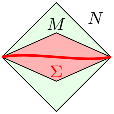

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- This is governed by the AQFT operad [Benini/AS/Woike, Benini/Carmona/AS]

$$\mathcal{O}_{(\mathbf{Loc}_m,\perp)}[\mathrm{Cauchy}^{-1}]^{\infty} \simeq \big(\mathcal{P}_{(\mathbf{Loc}_m,\perp)} \otimes \mathsf{uAs}\big)[\mathrm{Cauchy}^{-1}]^{\infty}$$

Classification in low dimensions (for target $\mathbf{T} =$ SM 1-category)

Prop: [Benini/AS/Woike] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \,\simeq\, \mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))} \quad ,$$

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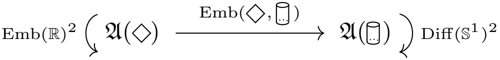

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$$\mathrm{Emb}(\mathbb{R})^2 \circlearrowright \mathfrak{A}(\Diamond) \xrightarrow{\mathrm{Emb}(\Diamond, \square)} \mathfrak{A}(\square) \circlearrowleft \mathrm{Diff}(\mathbb{S}^1)^2$$

- *Open problem:* Higher dimensions?

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\operatorname{char}\mathbb{K} = 0$)

- There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:
  (i) Strict time-slice axiom (projective model structure)

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$$L_! \,:\, \mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} \rightleftarrows \mathbf{AQFT}(\mathbf{C},\perp)^W \,:\, L^* \quad .$$

If L is a *reflective orthogonal localization*, then this is a Quillen equivalence.

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Rem: Very different behavior to topological QFTs (via locally constant factorization algebras on $\mathbb{R}^m$) $\rightsquigarrow$ $\mathbb{E}_m$-algebras [Lurie, Ayala/Francis]

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

- *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

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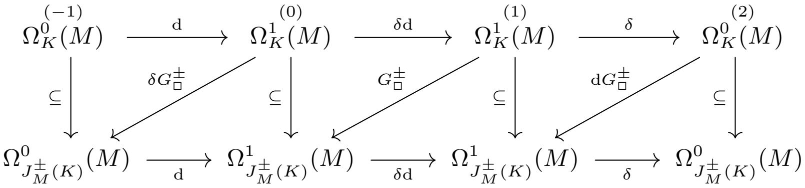

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\downarrow \subseteq & \delta G^{\pm}_{\square} & \downarrow \subseteq & G^{\pm}_{\square} & \downarrow \subseteq & \mathrm{d}G^{\pm}_{\square} & \downarrow \subseteq \\
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Thm: [Benini/Musante/AS] One can construct from such $\{\mathcal{F}(M), Q_M, \omega_M\}_{M \in \mathbf{Loc}_m}$ a $\mathbf{Ch}_{\mathbb{K}}$-valued AQFT $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_m, \perp)^{\mathrm{hoCauchy}}$.

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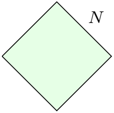

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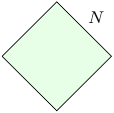

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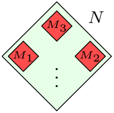

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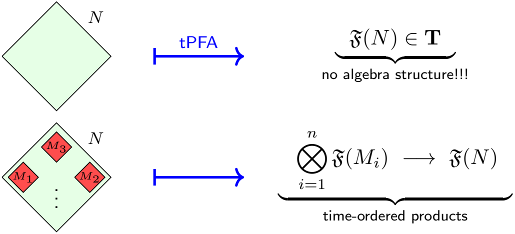

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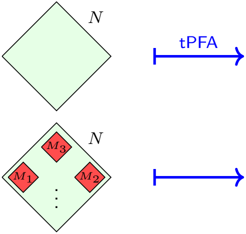

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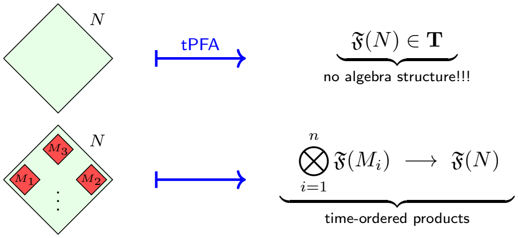

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◇ *Open problem:* Generalization to $\mathbf{T}$ = SM ∞-category, in particular $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$? In this case there are so far only example-based comparisons [Gwilliam/Rejzner, Benini/Musante/AS].

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  Work in progress [MacManus]: g.h. Lorentzian bordism double operads

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