Global derived geometry of lattice gauge fields

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Joint work with M. Benini and T. Fernández [arXiv:2409.06873].

Motivation and background

◇ A fundamental problem in (quantum) field theory is to describe moduli spaces of solutions to variational PDEs. Schematically:

$$\Big\{\text{fields } \Phi \text{ satisfying } \delta S(\Phi) = 0\Big\} \Big/ \Big\{\text{gauge symmetries } \Phi \overset{\sim}{\to} \Phi'\Big\}$$

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  - ② $\delta S(\Phi) = 0$ is typically not transverse $\Longrightarrow$ intersection singularities
  - ③ Fields Φ assemble into an ∞-dim. analytic object, e.g. Fréchet space
- A solution to these challenges is to refine the concept of geometry:

① $\rightsquigarrow$ stacks

① + ② $\rightsquigarrow$ derived stacks

① + ② + ③ $\rightsquigarrow$ some form of “derived analytic stacks”

Global vs. perturbative approaches

- ◇ Derived analytic geometry is currently in rapid development, e.g. by [Ben-Bassat/Kelly/Kremnitzer], [Clausen/Scholze] and [Steffens], but it remains a difficult topic that still has to find its way into field theory.

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- **Goal of this talk:**

  Provide a global/non-perturbative derived geometric description of classical GL_n-Yang-Mills theory on the 2-dimensional lattice $\mathbb{Z}^2$.

Derived algebraic geometry

◇ The building blocks of DAG (over $\mathbb{K}$ of char 0) are derived affine schemes

$$\mathbf{dAff} := \left(\mathbf{dgCAlg}^{\leq 0}\right)^{\mathrm{op}} \quad , \quad A^\bullet = \Big(\underbrace{\cdots \ A^{-2} \overset{\mathrm{d}}{\to} A^{-1} \overset{\mathrm{d}}{\to}}_{\substack{\text{higher structure refining} \\ \text{intersection constructions}}} A^0\Big)$$

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- Gluing derived affines w.r.t. étale hypercovers leads to derived stacks

$$\mathbf{dSt} := \mathbf{sSh}_{\text{ét}}\big(\mathbf{dAff}\big) \quad , \quad X(A^\bullet) = \Big(X^0(A^\bullet) \underbrace{\Leftarrow X^1(A^\bullet) \Lleftarrow X^2(A^\bullet) \;\cdots}_{\substack{\text{higher structure refining}\\ \text{quotient constructions}}}\Big)$$

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- ◇ A derived stack $X : \mathbf{dAff}^{\mathrm{op}} \to \mathbf{sSet}$ encodes derived structures (antifields) in its source and stacky structures (ghosts) in its target. This is at first a bit unusual when one is used to working perturbatively with L_∞-algebras.

Derived geometry in physics

- ◇ The link between DAG and (finite dim.) moduli problems from physics is:

  Given $X := \mathsf{Spec}\,A \in \mathbf{Aff} \subseteq \mathbf{dAff}$ ('fields') and $G := \mathsf{Spec}\,H \in \mathbf{Grp}(\mathbf{Aff})$ with action $X \times G \to X$ ('gauge symmetries'), consider the quotient stack

$$[X/G] \,:=\, \mathrm{hocolim}_{\mathbf{dSt}}\Big(X \Leftarrow X \times G \Lleftarrow X \times G^2 \,\cdots\Big) \in \mathbf{dSt} \quad .$$

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  For a function $S : [X/G] \longrightarrow \mathbb{A}^1$ ('action'), the space of critical points ('solutions to EL equations') is the derived critical locus

  $$\begin{tikzcd} \mathrm{dCrit}(S) \arrow[r, dashed] \arrow[d, dashed] & {[X/G]} \arrow[d, "\mathrm{d}_{\mathrm{dR}}S"] \\ {[X/G]} \arrow[r, "0"'] & T^*[X/G] \end{tikzcd}$$

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- ◇ The explicit description of such derived critical loci is typically difficult, but in the context above there are results.

Derived critical loci for quotient stacks

Thm: ([Benini/Safronov/AS], see also related [Anel/Calaque] and [Grataloup])

In the situation above where all the inputs are affine, the derived critical locus admits a presentation in terms of a derived quotient stack

$$\mathrm{dCrit}(S) \,\simeq\, [Z/G]$$

with $Z \in \mathbf{dAff}$ given by

$$\mathcal{O}(Z) \,=\, \mathrm{Sym}_A\Big((A \oplus \mathfrak{g}[2]) \oplus \mathsf{T}_A[1]\Big)$$

and the differential

$$\mathrm{d}a \,=\, 0 \quad , \quad \mathrm{d}v \,=\, \iota_v \mathrm{d}_{\mathrm{dR}} S \quad , \quad \mathrm{d}\xi \,=\, -\iota_{\rho(\xi)}\lambda \quad ,$$

for all $a \in A$, $v \in \mathsf{T}_A[1]$ and $\xi \in \mathfrak{g}[2]$, where λ is the tautological 1-form on T^*X and ρ is the induced $\mathfrak{g}$-action on T^*X.

The G-action on Z is induced by the G-actions on X and T^*X and the adjoint action on $\mathfrak{g}$.

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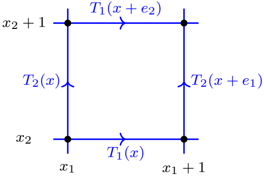

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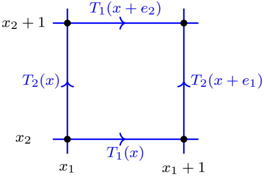

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$$\begin{aligned} \mathrm{Con}(\mathbb{Z}^2)\times\mathcal{G}(\mathbb{Z}^2) &\longrightarrow \mathrm{Con}(\mathbb{Z}^2)\\ \Big(\big(T_i(x)\big)_{(x,i)}, \big(U(x)\big)_x\Big) &\longmapsto \Big(U(x+e_i)^{-1}\,T_i(x)\,U(x)\Big)_{(x,i)} \end{aligned}$$

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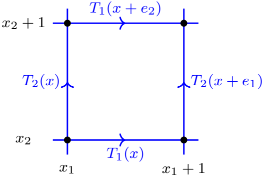

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Action: $S : \big[\mathrm{Con}(\mathbb{Z}^2)/\mathcal{G}(\mathbb{Z}^2)\big] \longrightarrow \mathbb{A}^1$ specified by

$$S(T) \,=\, \sum_{x\in\mathbb{Z}^2} \mathrm{Tr}\Big(\underbrace{T_2(x)^{-1}\,T_1(x+e_2)^{-1}\,T_2(x+e_1)\,T_1(x)}_{=:E(x)\ \text{ holonomy around plaquette}}\Big) \quad ,$$

called the Wilson action of lattice gauge theory.

Lattice gauge theory: Derived critical locus

- Applying the theorem from above, one obtains a model for the derived critical locus of GL_n-Yang-Mills theory on $\mathbb{Z}^2$ in terms of the quotient stack

$$\mathrm{dCrit}(S) \simeq \big[Z(\mathbb{Z}^2)/\mathcal{G}(\mathbb{Z}^2)\big]$$

where

$$\mathcal{O}\big(Z(\mathbb{Z}^2)\big) \cong \bigotimes_{x\in\mathbb{Z}^2} \mathrm{Sym}\big(\mathfrak{gl}_n[2]\big) \otimes \bigotimes_{(x,i)\in\mathbb{Z}^2\times\{1,2\}} \mathrm{Sym}\big(\mathfrak{gl}_n[1]\big) \otimes \bigotimes_{(x,i)\in\mathbb{Z}^2\times\{1,2\}} \mathcal{O}(\mathsf{GL}_n)$$

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with differential and $\mathcal{G}(\mathbb{Z}^2)$-action

$$\begin{aligned}
\mathrm{d}T_i(x) &= 0\\
\mathrm{d}\xi_1(x) &= E(x) - T_2(x-e_2)\,E(x-e_2)\,T_2(x-e_2)^{-1}\\
\mathrm{d}\xi_2(x) &= T_1(x-e_1)\,E(x-e_1)\,T_1(x-e_1)^{-1} - E(x)\\
\mathrm{d}\xi(x) &= -\xi_1(x) + T_1(x-e_1)\,\xi_1(x-e_1)\,T_1(x-e_1)^{-1}\\
&\quad -\xi_2(x) + T_2(x-e_2)\,\xi_2(x-e_2)\,T_2(x-e_2)^{-1}
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\rho\big(T_i(x)\big) &= U(x+e_i)^{-1}\,T_i(x)\,U(x)\\
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- Terminology: fields $T_i(x)$, antifields $\xi_i(x)$, ghosts $U(x)$ and antighosts $\xi(x)$.

Local derived critical loci

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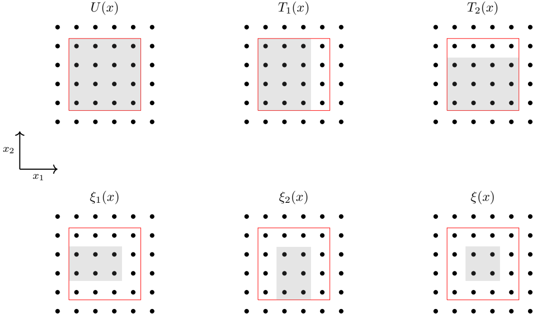
Def: For $V = [a,b] \times [c,d] \subseteq \mathbb{Z}^2$ a rectangular subset with side lengths $b - a \geq 2$ and $d - c \geq 2$, denote by $\mathcal{S}(V) = [Z(V)/\mathcal{G}(V)] \in \mathbf{dSt}$ the local derived critical locus generated by



Local constancy

Thm: ([Benini/Fernández/AS])
The local derived critical loci from above assemble into a functor

$$\mathcal{S} \,:\, \mathbf{Rect}(\mathbb{Z}^2)^{\mathrm{op}} \longrightarrow \mathbf{dSt}$$

which is locally constant in the sense that $\mathcal{S}(V') \xrightarrow{\sim} \mathcal{S}(V)$ is a weak equivalence, for every inclusion $V \subseteq V'$ of rectangular subsets.

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A key step in our proof strategy was to pass to weakly equivalent models $\mathcal{S} \simeq \mathcal{S}^{\mathrm{gf}_1} \simeq \mathcal{S}^{\mathrm{gf}_2}$ given by implementing an axial gauge-fixing $T_1(x) = \mathbb{1}$ (or $T_2(x) = \mathbb{1}$) for all $x \in \mathbb{Z}^2$.

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This allowed us to reduce the problem to a (still quite sophisticated) cohomology computation in $\mathbf{dgCAlg}^{\leq 0}$.

Observables

- Since the local derived critical loci $\mathcal{S}(V) = [Z(V)/\mathcal{G}(V)] \in \mathbf{dSt}$ are only 1-affine (instead of 0-affine), the appropriate observables

$$\mathfrak{F}(V) := \mathrm{QCoh}\big(\mathcal{S}(V)\big) \simeq {}_{\mathcal{O}(Z(V))}\mathbf{dgMod}_{\mathrm{cof}}^{\mathcal{O}(\mathcal{G}(V))} \in \mathbf{dgCat}$$

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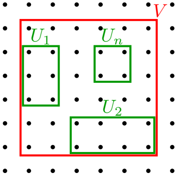
The classical observables of GL_n-Yang-Mills theory on $\mathbb{Z}^2$ form a locally constant prefactorization algebra $\mathfrak{F} \in \mathbf{Alg}_{\mathcal{P}_{\mathbb{Z}^2}}^{\mathrm{l.c.}}\big(\mathbf{dgCat}\big)$.



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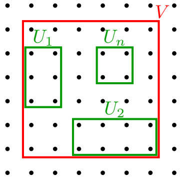

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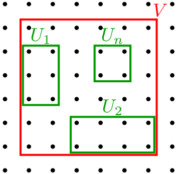

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⚠ **Open problems:** Shifted Poisson geometry and quantization? How does this lead to the well-known area dependence of $2d$ quantum Yang-Mills theory?