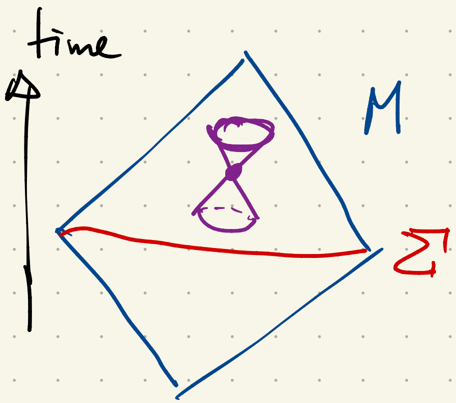
An AQFT perspective on quantum gauge theories

Alexander Schenkel (University of Nottingham)

Online talk @ Texas Tech, 22 March 2022

Based on joint works with M. Benini, S. Bruinsma, M. Perin, J. P. Pridham, P. Safronov, U. Schreiber and L. Woike

AQFT studies QFTs on Lorentzian manifolds:

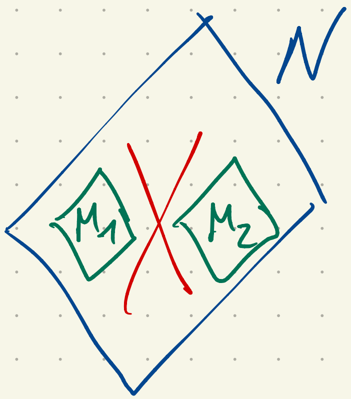


- A Lorentzian manifold is a manifold M with a metric of signature $(-++\cdots+)$.
- M is called globally hyperbolic if it admits a Cauchy surface $\Sigma \subset M$.
- Denote by $\mathbf{Loc}_m$ the category of oriented and time-oriented globally hyperbolic Lorentzian m-manifolds with morphisms the causal isometric embeddings.
- Important structures on $\mathbf{Loc}_m$:

"Orthogonality relation":

$$\begin{tikzcd} & N & \\ M_1 \arrow[ur, "f_1"] \arrow[rr, phantom, "\perp"] & & M_2 \arrow[ul, "f_2"'] \end{tikzcd}$$

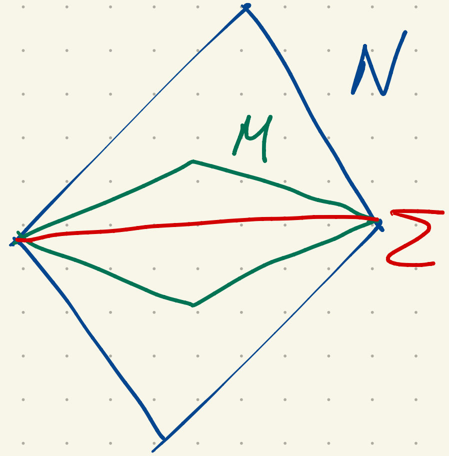
causally disjoint



"Weak equivalences":

$$f : M \xrightarrow{\sim} N$$

Cauchy morphism

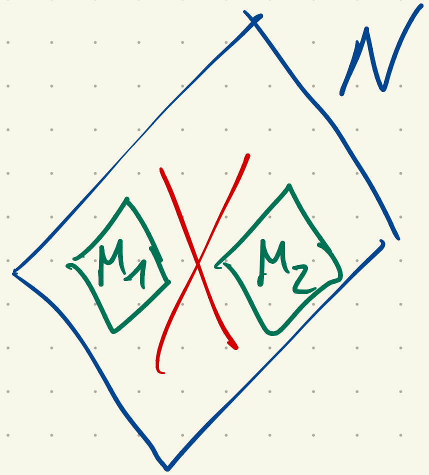


Traditional Definition: [Brunetti,Fredenhagen,Verch]

An m-dimensional AQFT is a functor $\mathfrak{A}: \mathbf{Loc}_m \longrightarrow {}^{(*)}\mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec}_{\mathbb{C}})$ that satisfies the following properties:

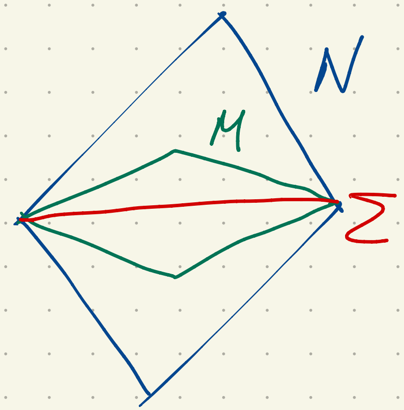
1.) **Einstein causality:** For all $(f_1: M_1 \to N) \perp (f_2: M_2 \to N)$,

$$\begin{tikzcd}
\mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \arrow[r, "\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)"] \arrow[d, "\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{A}(N) \otimes \mathfrak{A}(N) \arrow[d, "\mu_N^{op}"] \\
\mathfrak{A}(N) \otimes \mathfrak{A}(N) \arrow[r, "\mu_N"'] & \mathfrak{A}(N)
\end{tikzcd}$$



2.) **Time-slice:** For all $f: M \xrightarrow{\sim} N$,

$$\mathfrak{A}(f): \mathfrak{A}(M) \xrightarrow[\text{iso}]{\cong} \mathfrak{A}(N)$$



Operadic perspective:

- AQFTs are (*-) algebras over a colored (*-)operad $\mathcal{O}_{\overline{\mathrm{Loc}_m}}[W^{-1}]$ [Benini, AS, Woike].

- There exists an insightful presentation as the Boardman-Vogt tensor product

$$\mathcal{O}_{\overline{\mathrm{Loc}_m}}[W^{-1}] \simeq \boxed{\mathcal{P}_{\overline{\mathrm{Loc}_m}}}[W^{-1}] \underset{BV}{\otimes} \boxed{\mathsf{As}}$$

unital associative operad

with the Lorentzian prefactorization operad

$$\mathcal{P}_{\overline{\mathrm{Loc}_m}} = \begin{cases} \underline{\text{Obj:}} & M \in \mathrm{Loc}_m \\ \underline{\text{Mor:}} & \mathcal{P}\binom{N}{M_1 \dots M_n} = \left\{ (f_1, \dots, f_n) \in \prod_{i=1}^{n} \mathrm{Loc}_m(M_i, N) \; : \; f_i \perp f_j \;\; \forall i \neq j \right\} \end{cases}$$

- <u>Hence:</u>

$$\mathbf{AQFT}_m \simeq \mathbf{Alg}_{\mathcal{P}_{\overline{\mathrm{Loc}_m}}[W^{-1}]}\Big({}^{(*)}\mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec}_{\mathbb{C}}) \Big) = \left\{ \begin{array}{l} {}^{(*)}\mathbf{Alg}_{\mathsf{As}}\text{-valued Lorentzian analog} \\ \text{of prefactorization algebras, based} \\ \text{on causal disjointness } \perp \text{, that are} \\ \text{locally constant w.r.t. Cauchy morphisms} \end{array} \right.$$

Why As-algebras?

- The As-algebra $\mathcal{A}(M)$ should be interpreted as a quantization of the function algebra $\mathcal{O}(\mathrm{Sol}(M))$ of the solution space of a variational PDE.

- Key difference between Riemannian and Lorentzian QFT:

Riemannian	Lorentzian
$\mathrm{Sol}(M)$ is (-1)-shifted Poisson	$\mathrm{Sol}(M) \xrightarrow{\simeq} \mathrm{Data}(\Sigma)$ is unshifted Poisson via well-posed initial value problem
$\Rightarrow$ $\mathcal{O}(\mathrm{Sol}(M))$ is P_0-algebra	$\Rightarrow$ $\mathcal{O}(\mathrm{Sol}(M))$ is P_1-algebra
$\Rightarrow$ $\mathcal{O}_\hbar(\mathrm{Sol}(M))$ is E_0-algebra	$\Rightarrow$ $\mathcal{O}_\hbar(\mathrm{Sol}(M))$ is E_1-algebra $\simeq$ As-algebra

What about gauge theories?

$X : CDGA^{\leq 0} \longrightarrow sSet$

- In gauge theories, $Sol(M)$ is a derived stack, namely the derived critical locus of an action function $S_M : Fields(M) \longrightarrow \mathbb{R}$.

⚠ Such derived stacks are in general NOT affine, i.e. NOT determined by their function dg-algebra $\mathcal{O}(X) = \operatorname{holim}_{\operatorname{Spec} R \to X} R \in CDGA$.

- **Proposal from DAG:**

  Assign instead the SM dg-category $QCoh(X) = \operatorname{holim}_{\operatorname{Spec} R \to X} {}_R\mathrm{dgMod}$.

- That's indeed better: For G reductive group,

$$\mathcal{O}([pt/G]) \simeq C^\bullet(G, \mathbb{C}) \simeq \mathbb{C} = \mathcal{O}(pt) \qquad \text{doesn't see } G$$

$$QCoh([pt/G]) \simeq \mathrm{dgRep}_{\mathbb{C}}(G) \not\simeq Ch_{\mathbb{C}} = QCoh(pt) \qquad \text{sees } G$$

- Two pathways to quantization:

$$Sol(M) \rightsquigarrow \mathcal{O}_\hbar(Sol(M)) \quad \text{non-commutative dg-algebra}$$

$$Sol(M) \rightsquigarrow QCoh_\hbar(Sol(M)) \quad \text{pointed (i.e. } E_0\text{-monoidal) dg-category}$$

Two flavours of ∞-AQFT:

Affine ∞-AQFTs [Benini, AS, Woike]	∞-AQFTs [Benini, Perin, AS, Woike: 2AQFTs] [Benini, Pridham, AS: Toy models]
$\mathfrak{A} \in \mathrm{Alg}_{\mathcal{P}_{\overline{\mathrm{Loc}_m}}}(\mathrm{DGA}) \simeq \mathrm{Alg}_{\mathcal{O}_{\overline{\mathrm{Loc}_m}}}(\mathrm{Ch}_{\mathbb{C}})$	pointing comes from here! $\mathfrak{A} \in \mathrm{Alg}_{\mathcal{P}_{\overline{\mathrm{Loc}_m}}}(\mathrm{DGCat})$
satisfying homotopy time-slice	satisfying homotopy time-slice
$\mathfrak{A}(f): \mathfrak{A}(M) \xrightarrow[\text{w.e.}]{\sim} \mathfrak{A}(N)$	$\mathfrak{A}(f): \mathfrak{A}(M) \xrightarrow[\text{w.e.}]{\sim} \mathfrak{A}(N)$
for all $f: M \xrightarrow{\sim} N$ Cauchy	for all $f: M \xrightarrow{\sim} N$ Cauchy

• **Remarks:**

1.) Taking dg-modules defines a functor $(\text{Affine } \infty\text{-AQFTs}) \xrightarrow{(-)^{\mathrm{dgMod}}} (\infty\text{-AQFTs})$, which I believe is fully faithful. (We can prove this for 2AQFTs.)

2.) Perturbative QFTs are affine because formal moduli problems admit description in terms of their Chevalley–Eilenberg dg-algebras.

Constructing examples I: Linear Yang-Mills theory

- Linear (i.e. "free") QFTs can be constructed very explicitly [Benini, Bruinsma, AS]
- Consider the example of linear Yang-Mills theory:

$$\mathrm{Fields}(M) = \Big(\underset{\text{ghosts}}{\overset{(-1)}{\Omega^0(M)}} \xrightarrow{\ d\ } \underset{\text{fields}}{\overset{(0)}{\Omega^1(M)}} \Big) \quad \text{w/ action} \quad S_M = \int_M \underbrace{\tfrac{1}{2}\, dA \wedge * dA}_{\text{quadratic!}}$$

- Forming linear derived critical locus

$$\begin{tikzcd}
\mathrm{Sol}(M) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\text{homotopy pullback in } \mathrm{Ch}_{\mathbb{R}}"] & \mathrm{Fields}(M) \arrow[d, "0"] \\
\mathrm{Fields}(M) \arrow[r, "\delta S_M"'] & T^*\mathrm{Fields}(M)
\end{tikzcd}$$

gives

$$\mathrm{Sol}(M) = \Big(\underset{\text{ghosts}}{\overset{(-1)}{\Omega^0(M)}} \xrightarrow{\ d\ } \underset{\text{fields}}{\overset{(0)}{\Omega^1(M)}} \xrightarrow{\overset{\text{Maxwell operator}}{d*d}} \underset{\text{antifields}}{\overset{(1)}{\Omega^{m-1}(M)}} \xrightarrow{\ d\ } \underset{\text{antifields for ghosts}}{\overset{(2)}{\Omega^m(M)}} \Big)$$

- Via the integration pairing on forms, we can take function dg-algebra

$$\mathcal{O}(\mathrm{Sol}(M)) := \mathrm{Sym}\,\mathcal{L}(M) \qquad \text{with} \qquad \mathcal{L}(M) := \mathrm{Sol}_c(M)[1] .$$

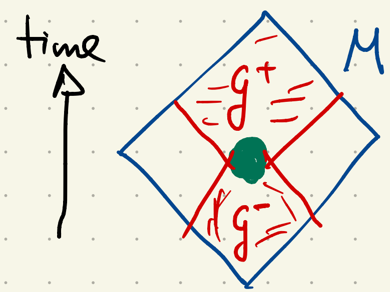
The shifted Poisson structure is given by extending to Sym the Ch-map

$$\Pi_M : \mathcal{L}(M) \otimes \mathcal{L}(M) \xrightarrow{\ \subseteq\ } \mathcal{L}(M) \otimes \mathrm{Sol}(M)[1] \xrightarrow{\text{integrate}} \mathbb{R}[1]$$

- Key observation: $\Pi_M = \partial(\text{something})$ is exact !

There exist (contractible spaces of) retarded / advanced Green homotopies

$$\begin{tikzcd} \mathcal{L}(M) \arrow[r, bend left=50, "0"] \arrow[r, phantom, shift left=3, "\Downarrow G^+"] \arrow[r, "\subseteq"] \arrow[r, phantom, shift right=3, "G^- \Uparrow"] \arrow[r, bend right=50, "0"'] & \mathrm{Sol}(M)[1] \end{tikzcd}$$



- This defines unshifted Poisson structure $\tau_M : \mathcal{L}(M) \otimes \mathcal{L}(M) \xrightarrow{\mathrm{id} \otimes (G^+ - G^-)} \mathcal{L}(M) \otimes \mathrm{Sol}(M) \xrightarrow{\int_M} \mathbb{R}$,

whose quantization yields affine ∞-AQFT: $\mathfrak{A}(M) = \dfrac{\mathrm{Free}\,\mathcal{L}(M)}{\langle \varphi\psi - (-1)^{|\varphi||\psi|}\psi\varphi - \hbar\,\tau_M(\varphi,\psi) \rangle}$

Constructing examples II: Non-Abelian Yang-Mills theory on a graph

- The derived non-Abelian Yang-Mills stack

$$\mathrm{Sol}^{YM}(M) = d\mathrm{Crit}\Big(S^{YM} : \underbrace{\mathrm{Con}_G(M)}_{=\{G\text{-bdl w/ connection on } M\}/\text{gauge equivalence}} \longrightarrow \mathbb{R} \Big)$$

is super complicated because it is an ∞-dimensional derived stack.

- So far we only understand two related (but much simpler) cases:

  (i) $d\mathrm{Crit}\big(f : [\mathrm{Spec}\,A / \mathrm{Spec}\,H] \longrightarrow \mathbb{K} \big)$ for a function on a finite-dimensional quotient stack [Benini, Safronov, AS]

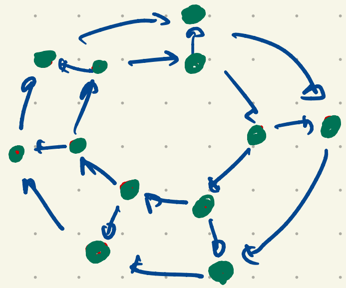
  (ii) The UNDERIVED Yang-Mills stack [Benini, Schreiber, AS]

- To get a feeling for non-Abelian quantum gauge theories, let us consider a lattice approximation to YM on a Cauchy surface $\Sigma \subset M$ [Benini, Pridham, AS]

- Approximate $\Sigma \subset M$ by a directed graph with vertices V and edges E:

$\Sigma \subseteq M$

approximate $\rightsquigarrow$



YM phase space on Σ $\rightsquigarrow$ finite-dimensional derived cotangent stack

$$T^*\mathrm{Con}_G(\Sigma) \rightsquigarrow T^*[\mathcal{E}(\Sigma)/\mathcal{G}(\Sigma)] = T^*\Big[\prod_{e\in E} G \Big/ \prod_{v\in V} G\Big]$$

- Using Pridham's theory of stacky CDGAs, one can work out quite explicitly the dg-category $\mathrm{QCoh}\big(T^*[\mathcal{E}(\Sigma)/\mathcal{G}(\Sigma)]\big)$ and its quantization $\mathfrak{A}(\Sigma)$ along the canonical unshifted Poisson structure. (See next slides...)

- These quantized dg-categories assemble into a lattice ∞-AQFT

$$\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{\mathrm{DiGraph}}}(\mathbf{DGCat})$$

Quantization of derived cotangent stacks:

The dg-category $QCoh_{\hbar}(T^*[\mathrm{Spec}\,A/\mathrm{Spec}\,H])$ is given by certain D-modules:

- An object is a triple $(\mathcal{E}_\bullet, \nabla, \Psi)$ consisting of:

    (1) an $A[[\hbar]]$-dg-module $\mathcal{E}_\bullet$ w/ coaction $\rho: \mathcal{E}_\bullet \to \mathcal{E}_\bullet \otimes H$

    (2) an H-equivariant $\hbar d^{dR}$-connection ∇ on $\mathcal{E}_\bullet$

    (3) an H-equivariant graded $A[[\hbar]]$-module map $\Psi: \mathfrak{g}_{[-1]} \otimes \mathcal{E}_\# \to \mathcal{E}_\#$

  satisfying the following properties:

    (i) $\nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar \nabla_{[v,v']}$, $\quad \nabla_v \Psi_t = \Psi_t \nabla_v$, $\quad \Psi_t \Psi_{t'} = -\Psi_{t'} \Psi_t$

    (ii) $\partial \Psi_t + \Psi_t \partial = \nabla_{\mu^*(t)} + \hbar\, \rho(t)$

- For G reductive, the Hom-complexes are H-equivariant D-module maps.

How was this obtained? VERY rough sketch!!!

$$T^*[X/G] \simeq [T^*X/\!/G] = [\mu^{-1}(0)/G]$$

symplectic reduction

$$[\mu^{-1}(0)/G] \leftarrow \cdots \left([\mu^{-1}(0)/\mathfrak{g}] \leftleftarrows [\mu^{-1}(0)\times G/\mathfrak{g}^2] \leftleftarrows \cdots \right)$$

resolution by stacky CDGAs

$$\mathrm{QCoh}(T^*[X/G]) \simeq \mathrm{holim}\left(\boxed{CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)))}\text{-dgMod} \rightrightarrows \boxed{CE^\bullet(\mathfrak{g}^2, \mathcal{O}(\mu^{-1}(0))\otimes H)}\text{-dgMod} \rightrightarrows \cdots \right)$$

$$\mathrm{QCoh}_\hbar(T^*[X/G]) := \mathrm{holim}\left(\boxed{\mathrm{DiffOp}_\hbar}\text{-dgMod} \rightrightarrows \boxed{\mathrm{DiffOp}_\hbar}\text{-dgMod} \rightrightarrows \cdots \right)$$