Integrable field theories in two and higher dimensions

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Joint work with M. Benini and B. Vicedo [arXiv:2601.19993]
and M. Benini, R. Cullinan and B. Vicedo [arXiv:2604.24864].

Motivation and background

◇ A field theory on a 2d spacetime Σ is Lax integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}[A, A] = 0$$

for a Lax connection $A = A_t\,\mathrm{d}t + A_x\,\mathrm{d}x \in \Omega^1(\Sigma, \mathfrak{g})\hat{\otimes}\,\mathcal{M}(C)$, constructed from the fields on Σ, depending meromorphically on a Riemann surface C.

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◇ **Important consequence:** The holonomy $\mathrm{Tr}\big(\mathrm{hol}_t(A)\big)$ along Cauchy surfaces $S_t \subseteq \Sigma$ is time-independent $\partial_t \mathrm{Tr}\big(\mathrm{hol}_t(A)\big) = 0$, so its Laurent expansion

$$\mathrm{Tr}\big(\mathrm{hol}_t(A)\big) \,=\, \sum_{n\in\mathbb{Z}} Q_n(A)\,z^n \qquad \text{on } C$$

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  - **Modern POV:** Gauge-theoretic methods explain origin of Lax connection ☺

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  Upon "integrating out" $\Sigma \times C \rightsquigarrow \Sigma$ the fields on the Riemann surface C, different choices provide different 2d IFTs and their Lax connections.
- **Goal of this talk:** Give a mathematical description of this story via L_∞-algebras and homotopy theory, and show how this scales to $\dim > 2$.

Quick recap: L_∞-algebras in field theory

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Ex:
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$$L_{\mathrm{scalar}} = \Big(\Omega^0(M) \xrightarrow{\ell_1 = \mathrm{d}*\mathrm{d}} \Omega^m(M) \Big)$$

with $\ell_n(\Phi_1,\dots,\Phi_n) = \lambda_n *(\Phi_1\cdots\Phi_n)$ and $\langle\!\langle \Phi, \Phi^\ddagger\rangle\!\rangle = \int_M \Phi\,\Phi^\ddagger$.

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- Chern-Simons $S(A) = \int_M \big\langle A, \frac{1}{2!}\mathrm{d}A + \frac{1}{3!}[A,A]\big\rangle$ is described by

$$L_{\mathrm{CS}} = \Big(\Omega^0(M,\mathfrak{g}) \xrightarrow{\ell_1=\mathrm{d}} \Omega^1(M,\mathfrak{g}) \xrightarrow{\ell_1=\mathrm{d}} \Omega^2(M,\mathfrak{g}) \xrightarrow{\ell_1=\mathrm{d}} \Omega^3(M,\mathfrak{g}) \Big)$$

  with $\ell_2(\alpha,\beta) = [\alpha,\beta]$ and $\langle\!\langle \alpha,\beta\rangle\!\rangle = \int_M \langle \alpha,\beta\rangle$.

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! Have to take into account singularities and boundary conditions at the zeros and poles of the meromorphic 1-form ω on C!

Divisors and holomorphic line bundles

- Poles and zeros on a Riemann surface C are best described by using divisors $D : C \to \mathbb{Z}$. The divisor $(\omega) = (\omega)_0 + (\omega)_\infty$ of ω encodes the locations of its zeros $(\omega)_0 \geq 0$ and of its poles $(\omega)_\infty \leq 0$.

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- A standard construction assigns to a divisor D a holomorphic line bundle $L_D \to C$ such that $\Gamma_{\mathrm{hol}}(L_D \to C) \cong \mathcal{O}_D(C) = \{f \in \mathcal{M}(C) \,:\, (f) \geq -D\}$ are meromorphic functions with poles and zeros controlled by D.

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- A key feature of divisors is that they alter Dolbeault cohomologies, e.g.:

$$H^0\Omega^{0,\bullet}(\mathbb{C}P^1, L_D) \cong \begin{cases} 0 & , \text{ if } \deg D < 0 \\ \mathbb{C}^{1+\deg D} & , \text{ if } \deg D \geq 0 \end{cases}$$

$$H^1\Omega^{0,\bullet}(\mathbb{C}P^1, L_D) \cong \begin{cases} \mathbb{C}^{-(1+\deg D)} & , \text{ if } \deg D < 0 \\ 0 & , \text{ if } \deg D \geq 0 \end{cases}$$

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- Poles and zeros on a Riemann surface C are best described by using divisors $D : C \to \mathbb{Z}$. The divisor $(\omega) = (\omega)_0 + (\omega)_\infty$ of ω encodes the locations of its zeros $(\omega)_0 \geq 0$ and of its poles $(\omega)_\infty \leq 0$.

- A standard construction assigns to a divisor D a holomorphic line bundle $L_D \to C$ such that $\Gamma_{\mathrm{hol}}(L_D \to C) \cong \mathcal{O}_D(C) = \{f \in \mathcal{M}(C) \,:\, (f) \geq -D\}$ are meromorphic functions with poles and zeros controlled by D.

- A key feature of divisors is that they alter Dolbeault cohomologies, e.g.:

$$H^0\Omega^{0,\bullet}(\mathbb{C}P^1, L_D) \cong \begin{cases} 0 & , \text{ if } \deg D < 0 \\ \mathbb{C}^{1+\deg D} & , \text{ if } \deg D \geq 0 \end{cases}$$

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! That's precisely where the non-trivial features of IFTs and Lax connections in the Costello-Yamazaki framework come from!

Singularity structures and Lax connections

Def: A singularity structure $\mathscr{D}$ for top.-hol. Chern-Simons theory $(\mathcal{E}(X), \ell)$ is a decomposition $(\omega)_0 = D_+ + D_-$ of the zeros of ω into two non-negative divisors $D_\pm \geq 0$.

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$$\mathcal{E}_{\mathscr{D}}(X) := \mathrm{Tot}\left(\Omega^{0,(0,\bullet)}(X,\mathfrak{g}) \begin{array}{c} \xrightarrow{\mathrm{d}_\Sigma^+} \\ \\ \xrightarrow[\mathrm{d}_\Sigma^-]{} \end{array} \begin{array}{c} \Omega^{+,(0,\bullet)}\big(X, L_{D_+}\otimes\mathfrak{g}\big) \\ \oplus \\ \Omega^{-,(0,\bullet)}\big(X, L_{D_-}\otimes\mathfrak{g}\big) \end{array} \begin{array}{c} \xrightarrow{\mathrm{d}_\Sigma} \\ \\ \xrightarrow[\mathrm{d}_\Sigma]{} \end{array} \Omega^{2,(0,\bullet)}\big(X, L_{(\omega)_0}\otimes\mathfrak{g}\big) \right)$$

and $\ell_2 = [\,\cdot\,,\,\cdot\,]$, where $\Omega^\pm(\Sigma) \subseteq \Omega^1(\Sigma)$ are the (a)sd 1-forms $*\alpha = \pm\alpha$.

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Prop: Through homotopy transfer to twisted-Dolbeault cohomologies, this is weakly equivalent to the L_∞-algebra given by

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and $\ell_2' = [\,\cdot\,,\,\cdot\,]$, describing Lax connections and their meromorphic behavior.

Boundary conditions and integrable field theories

Def: A boundary condition $\mathscr{B}$ for the $\mathscr{D}$-singular fields $(\mathcal{E}_{\mathscr{D}}(X), \ell)$ is a choice of non-positive divisors $B_0 \leq B_\pm \leq B_2 \leq 0$ satisfying

(i) $B_0 + B_2 = (\omega)_\infty = B_+ + B_-$

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Prop: Through homotopy transfer to twisted-Dolbeault cohomologies, this is weakly equivalent to the L_∞-algebra given by

$$\mathcal{F}(\Sigma) = \left(\Omega^0(\Sigma, \mathfrak{g}^N) \xrightarrow{\ell_1' = p_2\, \mathrm{d}_\Sigma (h_+ \mathrm{d}_\Sigma^+ + h_- \mathrm{d}_\Sigma^-) i_0} \Omega^2(\Sigma, \mathfrak{g}^N) \right)$$

with ℓ_n' for $n \geq 2$ determined by transfer and $N := -1 - \deg B_0 \geq 0$.

What can one say about the 2d field theory $(\mathcal{F}(\Sigma), \ell')$?

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1. The underlying complex resembles that of a $\mathfrak{g}^N$-valued scalar field, with $\ell'_1 = p_2\,\mathrm{d}_\Sigma(h_+\mathrm{d}^+_\Sigma + h_-\mathrm{d}^-_\Sigma) i_0$ given by a 2nd order differential operator.

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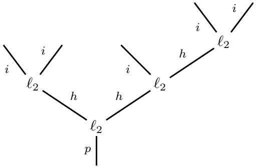

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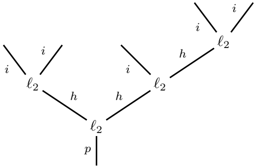

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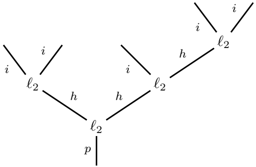

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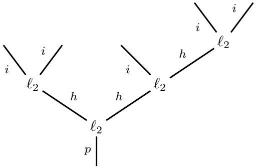

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Rem: With some computational effort (work in preparation), one can compute $(\mathcal{F}(\Sigma), \ell')$ explicitly for simple ω's and finds indeed a sigma-model.

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Thm: Combined with our weak equivalences, this yields an ∞-morphism

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! This result makes precise the slogan that "topological-holomorphic Chern-Simons theories give rise to both IFTs and their Lax connections".

Does this extend to higher dimensions?

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  1. Defining $(d+3)$-dimensional top.-hol. Chern-Simons theory on $M \times C$ requires a d-term L_∞-algebra $\mathfrak{g}$ with degree $d-1$ cyclic structure instead of an ordinary Lie algebra $\mathfrak{g}$ with invariant pairing $\langle \cdot , \cdot \rangle$.

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Open problem: Construct interesting examples of higher-dimensional IFTs and understand how they relate to models from physics.

A small glimpse at an example of a 3d IFT

- Applying the general machinery to $M = \mathbb{R}^3$ and a convenient choice of 2-term L_∞-algebra $\mathfrak{g} = T[1]\mathfrak{h}$, singularity structures and boundary conditions, one obtains a 3d IFT whose linearization is described by the complex:

$$\mathfrak{F}(M) = \left(\begin{array}{ccccccc} 0 & \xrightarrow{0} & \Omega^0(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^2(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M,\mathfrak{h}^2) \\ \oplus & & \oplus & & \oplus & & \oplus \\ \Omega^0(M,\mathfrak{h}^2) & \xrightarrow{\ell_1'} & \Omega^1(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M,\mathfrak{h}) & \xrightarrow{0} & 0 \end{array} \right)$$

$$\ell_1'(\Phi) = \begin{pmatrix} (q_1 - q_2)\,\partial_1\partial_2\Phi \\ (q_1 - q_3)\,\partial_1\partial_3\Phi \\ (q_2 - q_3)\,\partial_2\partial_3\Phi \end{pmatrix} \quad ,$$

$$\ell_1' \begin{pmatrix} \lambda_{12}^+ \\ \lambda_{13}^+ \\ \lambda_{23}^+ \end{pmatrix} = \begin{pmatrix} \partial_1\lambda_{23}^+ - \partial_2\lambda_{13}^+ + \partial_3\lambda_{12}^+ \\ -q_1\,\partial_1\lambda_{23}^+ + q_2\,\partial_2\lambda_{13}^+ - q_3\,\partial_3\lambda_{12}^+ \end{pmatrix} \quad ,$$

$$\ell_1' \begin{pmatrix} \epsilon \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\,\partial_1(\chi - q_1\,\epsilon) \\ -\frac{1}{2}\,\partial_2(\chi - q_2\,\epsilon) \\ -\frac{1}{2}\,\partial_3(\chi - q_3\,\epsilon) \end{pmatrix} \quad ,$$

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- Interestingly, this complex admits Green's operators associated with the d'Alembertian $\Box = \sum_{i,j=1}^3 g^{ij}\partial_i\partial_j$ of the Lorentzian metric

$$g^{-1} = \frac{1}{2}\begin{pmatrix} 0 & q_1-q_2 & q_1-q_3 \\ q_1-q_2 & 0 & q_2-q_3 \\ q_1-q_3 & q_2-q_3 & 0 \end{pmatrix}$$

obtained from the positions $q_1, q_2, q_3 \in \mathbb{R}$ of the zeros of ω.

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- Applying the general machinery to $M = \mathbb{R}^3$ and a convenient choice of 2-term L_∞-algebra $\mathfrak{g} = T[1]\mathfrak{h}$, singularity structures and boundary conditions, one obtains a 3d IFT whose linearization is described by the complex:

$$\mathfrak{F}(M) = \left(\begin{array}{ccccccc} 0 & \xrightarrow{0} & \Omega^0(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^2(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M,\mathfrak{h}^2) \\ \oplus & & \oplus & & \oplus & & \oplus \\ \Omega^0(M,\mathfrak{h}^2) & \xrightarrow{\ell_1'} & \Omega^1(M,\mathfrak{h}) & \xrightarrow{\ell_1'} & \Omega^3(M,\mathfrak{h}) & \xrightarrow{0} & 0 \end{array} \right)$$

$$\ell_1'(\Phi) = \begin{pmatrix} (q_1 - q_2)\,\partial_1\partial_2\Phi \\ (q_1 - q_3)\,\partial_1\partial_3\Phi \\ (q_2 - q_3)\,\partial_2\partial_3\Phi \end{pmatrix} \quad ,$$

$$\ell_1' \begin{pmatrix} \lambda_{12}^+ \\ \lambda_{13}^+ \\ \lambda_{23}^+ \end{pmatrix} = \begin{pmatrix} \partial_1\lambda_{23}^+ - \partial_2\lambda_{13}^+ + \partial_3\lambda_{12}^+ \\ -q_1\,\partial_1\lambda_{23}^+ + q_2\,\partial_2\lambda_{13}^+ - q_3\,\partial_3\lambda_{12}^+ \end{pmatrix} \quad ,$$

$$\ell_1' \begin{pmatrix} \epsilon \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\,\partial_1(\chi - q_1\,\epsilon) \\ -\frac{1}{2}\,\partial_2(\chi - q_2\,\epsilon) \\ -\frac{1}{2}\,\partial_3(\chi - q_3\,\epsilon) \end{pmatrix} \quad ,$$

$$\ell_1' \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = (q_1 - q_2)\,\partial_1\partial_2\lambda_3 - (q_1 - q_3)\,\partial_1\partial_3\lambda_2 + (q_2 - q_3)\,\partial_2\partial_3\lambda_1 \quad ,$$

- Interestingly, this complex admits Green's operators associated with the d'Alembertian $\Box = \sum_{i,j=1}^3 g^{ij}\partial_i\partial_j$ of the Lorentzian metric

$$g^{-1} = \frac{1}{2}\begin{pmatrix} 0 & q_1 - q_2 & q_1 - q_3 \\ q_1 - q_2 & 0 & q_2 - q_3 \\ q_1 - q_3 & q_2 - q_3 & 0 \end{pmatrix}$$

obtained from the positions $q_1, q_2, q_3 \in \mathbb{R}$ of the zeros of ω.

- This example shares some similarities with Ward's 3d IFT, and it would be interesting to carry out a detailed analysis of the interacting model.