Categorical techniques for NC geometry and gravity

Alexander Schenkel

School of Mathematical Sciences, University of Nottingham

Talk @ Mathematical models for NC in physics and quantum spacetime,
Banach Center, Warsaw, 2. – 3. November 2017.

Based joint works with G. E. Barnes and R. J. Szabo [1409.6331; 1507.02792; 1601.07353] and with P. Aschieri [1210.0241].

Outline

1. Background and motivation
2. Doing algebra in braided monoidal categories
3. Braided derivations and differential calculi
4. Connections and their lifts to tensor products
5. Towards NC vielbein gravity

Background and motivation

Recap: Connections on modules

- Let A be NC algebra and $(\Omega^\bullet, \mathrm{d})$ differential calculus over A, i.e.

$$\Omega^\bullet = \bigoplus_{n\geq 0} \Omega^n \qquad \text{with} \qquad \Omega^0 = A$$

  and d satisfies graded Leibniz rule

$$\mathrm{d}(\omega\,\omega') = (\mathrm{d}\omega)\,\omega' + (-1)^{|\omega|}\,\omega\,(\mathrm{d}\omega')$$

- A connection on a right A-module V is a linear map $\nabla : V \to V \otimes_A \Omega^1$ satisfying the right Leibniz rule

$$\nabla(v\,a) = \nabla(v)\,a + v \otimes_A \mathrm{d}a$$

- The set of connections $\mathrm{Con}(V)$ is affine space over $\mathrm{Hom}_A(V, V \otimes_A \Omega^1)$.

Connections on bimodules

◇ Let now V be an A-bimodule. What are connections on V?

◇ **Usual approach:** Bimodule connections [Mourad,Dubois-Violette,Masson,...]

A right module connection $\nabla : V \to V \otimes_A \Omega^1$ together with an A-bimodule homomorphism $\sigma : \Omega^1 \otimes_A V \to V \otimes_A \Omega^1$ such that

$$\nabla(a\,v) = a\,\nabla(v) + \sigma(\mathrm{d}a \otimes_A v) \qquad (\sigma\text{-twisted left Leibniz rule})$$

✓ Bimodule connections lift to tensor products:

Given (∇, σ) on V and (∇', σ') on V', then

$$\widetilde{\nabla} := (\mathrm{id}_V \otimes_A \sigma')\,(\nabla \otimes_A \mathrm{id}_{V'}) + \mathrm{id}_V \otimes_A \nabla' \,:\, V \otimes_A V' \longrightarrow V \otimes_A V' \otimes_A \Omega^1$$
$$\widetilde{\sigma} := (\mathrm{id}_V \otimes_A \sigma')\,(\sigma \otimes_A \mathrm{id}_{V'}) \,:\, \Omega^1 \otimes_A V \otimes_A V' \longrightarrow V \otimes_A V' \otimes_A \Omega^1$$

defines bimodule connection $(\widetilde{\nabla}, \widetilde{\sigma})$ on $V \otimes_A V'$.

ϟ Bimodule connections form an affine space over ${}_A\mathrm{Hom}_A(V, V \otimes_A \Omega^1)$.

$\rightsquigarrow$ such spaces are in general very small!

${}_A\mathrm{Hom}_A(V,W)$ is very small

- For simplicity, consider free A-bimodules $V = A^n$ and $W = A^m$.
- Right A-module homomorphisms are A-valued matrices

$$\mathrm{Hom}_A(V,W) \;\cong\; A^{m\times n} \ni L \quad : \quad (v_i)_{i=1}^n \;\longmapsto\; \Big(\sum_{i=1}^n L_{ki}\, v_i\Big)_{k=1}^m$$

- Such L's are A-bimodule homomorphisms iff all entries are central, i.e.

$${}_A\mathrm{Hom}_A(V,W) \;\cong\; Z(A)^{m\times n}$$

- **Example:** Let $A = \mathbb{C}[x^a, p_b]\big/(x^a p_b - p_b x^a - i\,\delta^a_b)$ be $2k$-dim. Moyal-Weyl algebra with standard differential calculus $\Omega^1 \cong A^{2k} \ni \omega_a\,\mathrm{d}x^a + \eta^b\,\mathrm{d}p_b$.

  The center is $Z(A) \cong \mathbb{C}$, hence:

  ! Bimodule connections on $V = A^n$ are affine space over the finite-dimensional vector space ${}_A\mathrm{Hom}_A(V, V\otimes_A \Omega^1) \;\cong\; \mathbb{C}^{2k\,n^2}$

- That's too rigid, in particular for applications to NC field and gravity theories.

How can we solve or avoid this issue?

☹ For generic NC algebras A, the concept of bimodule connections is what is needed for liftings to tensor products [Bresser,Müller-Hoissen,Dimakis,Sitarz].

☺ I will show that for "special" NC algebras, one can loosen the concept of bimodule connections and still obtains liftings to tensor products.

◇ Let me give a first hint what I mean by "special" by an example:

- Consider again the Moyal-Weyl algebra $A = \mathbb{C}[x,p]/(xp - px - i)$
- The product $\mu : A \otimes A \to A$ is clearly noncommutative

$$[a,b] = ab - ba = (\mu - \mu \circ \mathrm{flip})(a \otimes b) \neq 0$$

  when we use $\mathrm{flip} : A \otimes A \to A \otimes A\,,~ a \otimes b \mapsto b \otimes a$.

- However, using the nontrivial braiding

$$\tau : A \otimes A \longrightarrow A \otimes A~, \quad a \otimes b \longmapsto e^{i(\partial_p \otimes \partial_x - \partial_x \otimes \partial_p)} b \otimes a$$

  one finds that

$$[a,b]_\tau = (\mu - \mu \circ \tau)(a \otimes b) = 0 \quad \Rightarrow \quad \text{braided commutative!}$$

Doing algebra in braided monoidal categories

Recap: Monoidal categories and monoid objects

◇ A monoidal category is the following data:

- a category C,
- a functor $\otimes : C \times C \to C$ (tensor product),
- an object $I \in C$ (unit object),
- natural isomorphisms (associator and left/right unitor)

$$\alpha : (c_1 \otimes c_2) \otimes c_3 \;\cong\; c_1 \otimes (c_2 \otimes c_3) \quad , \quad \lambda : I \otimes c \;\cong\; c \quad , \quad \rho : c \otimes I \;\cong\; c$$

satisfying the pentagon and triangle identities.

◇ Internal to monoidal categories, one can talk about monoids:

A monoid (or algebra) in C is an object $A \in C$ together with C-morphisms $\mu : A \otimes A \to A$ (product) and $\eta : I \to A$ (unit) satisfying

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}\otimes\mu} & A \otimes A \\ {\scriptstyle \mu\otimes\mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ A \otimes A & & \xrightarrow[\mu]{\hspace{4em}} & & A \otimes A \end{array} \qquad \begin{array}{ccccc} I \otimes A & \xrightarrow{\eta\otimes\mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id}\otimes\eta} & A \otimes I \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \rho} & \\ & & A & & \end{array}$$

Ex: Monoids in $\mathrm{Vec}_{\mathbb{K}}$ are associative and unital $\mathbb{K}$-algebras.

Braided monoidal categories

- A braided monoidal category is a monoidal cat C with nat. iso (braiding)

$$\tau : c_1 \otimes c_2 \;\cong\; c_2 \otimes c_1$$

satisfying the hexagon identities. (If $\tau^2 = \mathrm{id}$, symmetric monoidal category.)

- This allows us to talk about braided commutative monoids/algebras:

A monoid (A, μ, η) in C is called braided commutative if the product is compatible with the braiding, i.e.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\;\tau\;} & A \otimes A \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & A & \end{array}$$

Rem: I like to interpret braided commutative monoids (A, μ, η) as algebras where the commutation relations are dictated by τ.

$\rightsquigarrow$ cf. Giovanni Landi's talk!

Bimodule objects

- Let C monoidal category and (A, μ, η) monoid in C.

- An A-bimodule in C is an object $V \in C$ together with C-morphisms $l : A \otimes V \to V$ (left action) and $r : V \otimes A \to V$ (right action) satisfying the obvious compatibilities.

- If C has coequalizers, we can equip the category ${}_A C_A$ of A-bimodules in C with a monoidal structure where $I_A = A$ and $\otimes_A$ is given by

$$(V \otimes A) \otimes W \underset{(\mathrm{id} \otimes l) \circ \alpha}{\overset{r \otimes \mathrm{id}}{\rightrightarrows}} V \otimes W \longrightarrow V \otimes_A W$$

- If C is braided monoidal category and A braided commutative monoid, we call $V \in {}_A C_A$ symmetric iff

$$\begin{array}{ccc} V \otimes A & \xrightarrow{\ \tau\ } & A \otimes V \\ & {\scriptstyle r} \searrow \quad \swarrow {\scriptstyle l} & \\ & V & \end{array} \qquad \textbf{and} \qquad \begin{array}{ccc} A \otimes V & \xrightarrow{\ \tau\ } & V \otimes A \\ & {\scriptstyle l} \searrow \quad \swarrow {\scriptstyle r} & \\ & V & \end{array}$$

Prop: The braiding on C descends to a braiding on the monoidal category ${}_A C_A^{\mathrm{sym}}$.

Examples from quasi-triangular Hopf algebras

- Let (H, R) be a quasi-triangular Hopf algebra.
- A left H-module is a vector space V with H-action $\triangleright : H \otimes V \to V$.
- Tensor products of left H-modules defines monoidal category $({}^H\mathscr{M}, \otimes, \mathbb{K})$.

  NB: Associators and unitors are trivial, hence they will be suppressed.
- Using R-matrix $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$, we obtain braiding

$$\tau : V \otimes W \longrightarrow W \otimes V \ , \quad v \otimes w \longmapsto R^{(2)} \triangleright w \otimes R^{(1)} \triangleright v$$

- Braided commutative monoids (A, μ, η) in ${}^H\mathscr{M}$ are H-module algebras satisfying the commutation relations $a\,b = (R^{(2)} \triangleright b)\,(R^{(1)} \triangleright a)$.

Ex: The Moyal-Weyl algebra, NC torus, Connes-Landi sphere, etc., are braided commutative for Hopf algebra $H = U\mathbb{R}^{2k}$ with $R = \exp(i\,\Theta^{lm} t_l \otimes t_m)$.

(More fancy example, see blackboard!)

- Associated to each braided commutative monoid (A, μ, η) in ${}^H\mathscr{M}$ is a braided monoidal category $({}^H_A\mathscr{M}^{\mathrm{sym}}_A, \otimes_A, A, \tau_A)$ of symmetric A-bimodules.

NB: For simplicity, I will focus for the rest of this talk on these examples!

Braided derivations and differential calculi

Ordinary vs. braided derivations: A first look

◇ An ordinary derivation on a braided commutative monoid A in ${}^H\mathscr{M}$ is an ${}^H\mathscr{M}$-morphism $X : A \to A$ satisfying the Leibniz rule

$$X(a\, b) = X(a)\, b + a\, X(b)$$

◇ A braided derivation on A is a linear map $X : A \to A$ (not necessarily H-equivariant!) satisfying the braided Leibniz rule

$$X(a\, b) = X(a)\, b + (R^{(2)} \triangleright a)\ (R^{(1)} \triangleright X)(b)$$

where the H-action on linear maps is via adjoint action

$$h \triangleright X := (h_{(1)} \triangleright \cdot\,) \circ X \circ (S(h_{(2)}) \triangleright \cdot\,)$$

Prop: There is a linear isomorphism

$$\big\{\text{ordinary derivations on } A\big\} \;\cong\; \big\{H\text{-invariant braided derivations on } A\big\}$$

$\Rightarrow$ There are many more braided derivations than ordinary ones! Hence, braided derivations are more flexible for doing geometry on A.

Categorical interpretation via internal homs

- ${}^H\mathscr{M}$ has internal homs $\zeta\,:\, \mathrm{Hom}_{{}^H\mathscr{M}}(Z\otimes V,W)\cong \mathrm{Hom}_{{}^H\mathscr{M}}(Z,\mathrm{hom}(V,W))$
- **Explicitly:** $\mathrm{hom}(V,W)\in {}^H\mathscr{M}$ is $\mathrm{Hom}_{\mathbb{K}}(V,W)$ with adjoint H-action.
- From abstract non-sense, one obtains ${}^H\mathscr{M}$-morphisms:
  - Evaluation: $\mathrm{ev}:\mathrm{hom}(V,W)\otimes V\to W$
  - Composition: $\bullet:\mathrm{hom}(W,Z)\otimes\mathrm{hom}(V,W)\to\mathrm{hom}(V,Z)$
- Let us adjoin $\mu:A\otimes A\to A$ to $\zeta(\mu):A\to\mathrm{end}(A)$ and define the bracket $[\,\cdot\,,\,\cdot\,]:=\bullet-\bullet\circ\tau:\mathrm{end}(A)\otimes\mathrm{end}(A)\to\mathrm{end}(A)$.
- Braided derivations on A are those $X\in\mathrm{end}(A)$ satisfying

$$[X,\zeta(\mu)(a)] \;=\; \zeta(\mu)\big(\mathrm{ev}(X\otimes a)\big)$$

- Formally, $\mathrm{der}(A)\in{}^H\mathscr{M}$ is characterized by the equalizer in ${}^H\mathscr{M}$

$$\mathrm{der}(A)\longrightarrow\mathrm{end}(A)\ \underset{\zeta(\zeta(\mu)\circ\mathrm{ev})}{\overset{\zeta([\,\cdot\,,\zeta(\mu)(\,\cdot\,)])}{\rightrightarrows}}\ \mathrm{hom}(A,\mathrm{end}(A))$$

Construction of Kähler-style differentials

- Similarly, one defines braided derivations $\mathrm{der}(A,V)$ valued in $V \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$
- One can show that $\mathrm{der}(A,V) \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$ via

$$(a \cdot X)(b) := a\,X(b) \quad , \qquad (X \cdot a)(b) := X\big(R^{(2)} \triangleright b\big)\,(R^{(1)} \triangleright a)$$

- The functor $\mathrm{der}(A,-) : {}^H_A\mathscr{M}^{\mathrm{sym}}_A \to {}^H_A\mathscr{M}^{\mathrm{sym}}_A$ is representable via

$$\mathrm{der}(A,-) \;\cong\; \mathrm{hom}_A(\Omega^1,-)$$

  where

  - $\mathrm{hom}_A(V,W) \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$ is internal hom in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$;
    (Right A-linear maps with adjoint H-action and suitable A-bimodule structure.)
  - $\Omega^1 = \Gamma/\Gamma^2 \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$ where $\Gamma := \ker\big(\mu : A \otimes A \to A\big)$ is H-module algebra.
    (The differential on Ω^1 is the typical one $\mathrm{d} : A \to \Omega^1\,,\; a \mapsto a \otimes 1 - 1 \otimes a$.)

- Construct $\Omega^\bullet$ as semifree braided graded-commutative DGA over Ω^1.
- **Conclusion:** Any braided commutative monoid A in ${}^H\mathscr{M}$ admits a canonical differential calculus obtained from braided derivations.

Connections and their lifts to tensor products

Connections on ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ via internal homs

- **Wanted:** "Carving out" space of connections on V from $\mathrm{hom}(V, V \otimes_A \Omega^1)$.
- First observe that the right Leibniz rule $\nabla(v\,a) = \nabla(v)\,a + v \otimes_A \mathrm{d}a$ is equivalent to (with $R^{-1} = \bar{R}^{(1)} \otimes \bar{R}^{(2)}$ inverse R-matrix)

$$\nabla(a\,v) - (R^{(2)} \triangleright a)\;(R^{(1)} \triangleright \nabla)(v) \;=\; \bar{R}^{(1)} \triangleright v \otimes_A \bar{R}^{(2)} \triangleright (\mathrm{d}a)$$

- Both sides are obtained from ${}^H\mathscr{M}$-morphisms to internal homs:
  - $\mathrm{lhs} := [\,\cdot\,, \zeta(l)(\,\cdot\,)] : \mathrm{hom}(V, V \otimes_A \Omega^1) \otimes A \to \mathrm{hom}(V, V \otimes_A \Omega^1)$
  - $\mathrm{rhs} := \zeta\big((\mathrm{id} \otimes \mathrm{d}) \circ \tau^{-1}\big) : A \to \mathrm{hom}(V, V \otimes_A \Omega^1)$ is the adjoint of the ${}^H\mathscr{M}$-morphism $(\mathrm{id} \otimes \mathrm{d}) \circ \tau^{-1} : A \otimes V \to V \otimes_A \Omega^1$
- Define the object of connections $\mathrm{con}(V) \in {}^H\mathscr{M}$ by equalizer

$$\mathrm{con}(V) \longrightarrow \mathrm{hom}(V, V \otimes_A \Omega^1) \times \mathbb{K} \underset{\mathrm{rhs} \circ \mathrm{pr}_2}{\overset{\mathrm{lhs} \circ \mathrm{pr}_1}{\rightrightarrows}} \mathrm{hom}\big(A, \mathrm{hom}(V, V \otimes_A \Omega^1)\big)$$

NB: Elements of $\mathrm{con}(V)$ are pairs $(\nabla, c) \in \mathrm{hom}(V, V \otimes_A \Omega^1) \times \mathbb{K}$ satisfying the "continuous" right Leibniz rule

$$\nabla(v\,a) = \nabla(v)\,a + c\,v \otimes_A \mathrm{d}a$$

Construction of tensor product connections

- **Question:** Given $V, V' \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$, $(\nabla, c) \in \mathrm{con}(V)$ and $(\nabla', c') \in \mathrm{con}(V')$. Can we construct a connection on $V \otimes_A V'$?

- That's indeed possible! To formalize our construction, we use:
  - Tensor product: $\circledast : \mathrm{hom}(V, W) \otimes \mathrm{hom}(X, Y) \to \mathrm{hom}(V \otimes X, W \otimes Y)$
  - Fiber product:

$$
\begin{array}{ccc}
\mathrm{con}(V) \times_{\mathbb{K}} \mathrm{con}(V') & \longrightarrow & \mathrm{con}(V') \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathbb{K}}} \\
\mathrm{con}(V) & \xrightarrow[\mathrm{pr}_{\mathbb{K}}]{} & \mathbb{K}
\end{array}
$$

Main Theorem [Barnes,AS,Szabo: 1507.02792]

Let $V, V' \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$. There exists an ${}^H\mathscr{M}$-morphisms (called sum of connections)

$$
\boxdot : \mathrm{con}(V) \times_{\mathbb{K}} \mathrm{con}(V') \longrightarrow \mathrm{con}(V \otimes_A V') \quad ,
$$

$$
\big((\nabla, c), (\nabla', c)\big) \longmapsto \Big(\tau_{23}\big(\nabla \circledast 1\big) + 1 \circledast \nabla', c\Big)
$$

Moreover, the sum of connections is associative.

Comparison to bimodule connections

How is our notion of connections different from bimodule connections?

Braided connections	**Bimodule connections**
$\nabla(v\,a) = \nabla(v)\,a + v \otimes_A \mathrm{d}a$	$\nabla(v\,a) = \nabla(v)\,a + v \otimes_A \mathrm{d}a$
$\nabla(a\,v) = (R^{(2)} \triangleright a)\,(R^{(1)} \triangleright \nabla)(v) + \bar{R}^{(1)} \triangleright v \otimes_A \bar{R}^{(2)} \triangleright (\mathrm{d}a)$	$\nabla(a\,v) = a\,\nabla(v) + \sigma(\mathrm{d}a \otimes_A v)$
$\nabla \boxdot \nabla'(v \otimes_A v') = \tau_{23}\big(\nabla(v) \otimes_A v'\big) + (R^{(2)} \triangleright v) \otimes_A (R^{(1)} \triangleright \nabla')(v')$	$\nabla \boxplus \nabla'(v \otimes_A v') = \sigma'_{23}\big(\nabla(v) \otimes_A v'\big) + v \otimes_A \nabla'(v')$

Further aspects of connections [Barnes,AS,Szabo: 1507.02792]

◇ To any connection $(\nabla, 1) \in \mathrm{con}(V)$ one can assign its curvature

$$\mathrm{curv}(\nabla) \in \mathrm{hom}_A(V, V \otimes_A \Omega^2)$$

◇ The curvature behaves additively under sums of connections

$$\mathrm{curv}\big(\nabla \boxplus \nabla'\big) = \tau_{23}\big(\mathrm{curv}(\nabla) \circledast 1\big) + 1 \circledast \mathrm{curv}(\nabla')$$

◇ Interpreting internal homs $\mathrm{hom}_A(V, W) \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$ as 'homomorphism bundles', we also would like to induce connections on them:

Thm: Let $V, W \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$. There exists an ${}^H\mathscr{M}$-morphism

$$\mathrm{ad}_\bullet : \mathrm{con}(W) \times_{\mathbb{K}} \mathrm{con}(V) \longrightarrow \mathrm{con}(\mathrm{hom}_A(V, W))$$

Cor: Denote by $V^\vee := \mathrm{hom}_A(V, A)$ the dual module. Then there exists an ${}^H\mathscr{M}$-morphism

$$(-)^\vee : \mathrm{con}(V) \longrightarrow \mathrm{con}(V^\vee)$$

Towards NC vielbein gravity

Deformation of vielbein gravity

- Let M be $4d$ spin mnf with trivial Dirac spinor bundle $S = M \times \mathbb{C}^4 \to M$
- Let $H = U\mathrm{Vec}(M)_F$ and $R = F_{21}\, F^{-1}$ for some cocycle twist F
- Twist deformation quantization constructs $\star$-product on $C^\infty(M)$ and $\star$-bimodule structure on $\Gamma^\infty(S)$, such that
  - $A = (C^\infty(M), \mu_F, \eta_F)$ is braided commutative monoid in ${}^H\mathscr{M}$;
  - $V = (\Gamma^\infty(S), l_F, r_F) \in {}^H_A\mathscr{M}^{\mathrm{sym}}_A$. (Note that $V \cong A^4$ is free.)
- NC vielbein gravity coupled to Dirac fields requires the following fields:
  - Dirac and co-Dirac field: $\psi \in V$ and $\overline{\psi} \in V^\vee = \mathrm{hom}_A(V, A)$;
  - Spin connection: $(\nabla, 1) \in \mathrm{con}(V)$ such that $\nabla = \mathrm{d} - \frac{1}{2}\omega^{ab}[\gamma_a\gamma_b]$;
  - Vielbein: $E \in \mathrm{end}_A(V)$ such that $E = E^a\,\gamma_a$.
- Using our constructions, we can define Lagrangian for this NC field theory

$$L = \mathrm{tr}\Big(i\,\mathrm{curv}(\nabla) \bullet E \bullet E \bullet \gamma_5 - \big(\nabla(\psi) \otimes_A \overline{\psi} - \psi \otimes_A \nabla^\vee(\overline{\psi})\big) \bullet E \bullet E \bullet E \bullet \gamma_5\Big)$$

NB: The noncommutativity is in the composition and evaluation of internal homs!