Higher Structures in Algebraic Quantum Field Theory

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Based on joint works with **M. Benini** and different subsets of

$\Big\{$ **S. Bruinsma, M. Perin, U. Schreiber, R. J. Szabo, L. Woike** $\Big\}$

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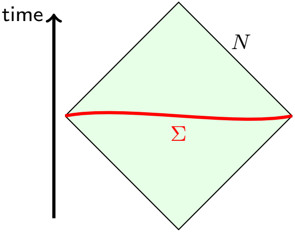
2. Explain why gauge theory requires higher categorical structures
3. Survey of our homotopical AQFT program for quantum gauge theories

Algebraic quantum field theory

What is an AQFT?

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N



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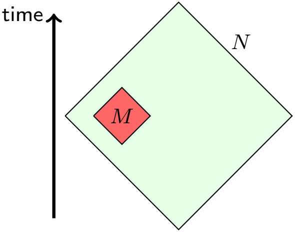

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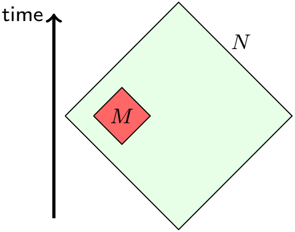

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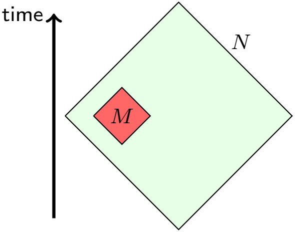

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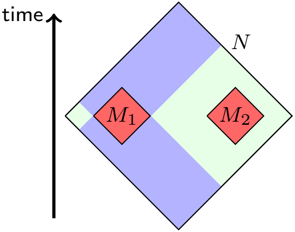

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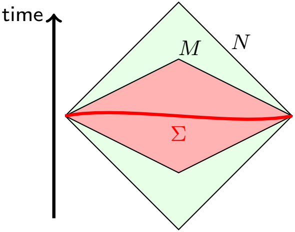

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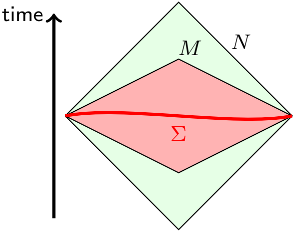

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$\rightsquigarrow$ AQFT links Lorentzian geometry to algebraic structure

AQFTs and colored operads

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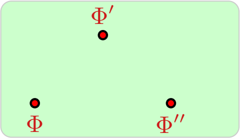
Ex: Linear Klein-Gordon theory is j-local (uses also results by [Lang])

Higher structures in gauge theory

What is a gauge theory?

"Ordinary" field theory:

Set/Space of fields



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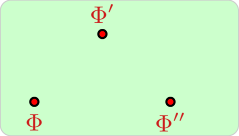

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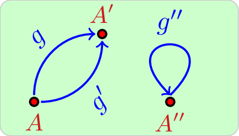

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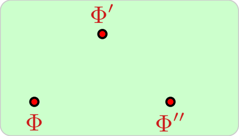

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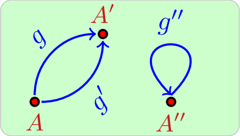

Ex: Groupoid of principal G-bundles with connection on $U \cong \mathbb{R}^m$

$$BG_{\mathrm{con}}(U) \;=\; \begin{cases} \underline{\text{Obj:}} & A \in \Omega^1(U, \mathfrak{g}) \\ \underline{\text{Mor:}} & A \xrightarrow{g \in C^\infty(U,G)} A \triangleleft g = g^{-1} A g + g^{-1} \mathrm{d} g \end{cases}$$

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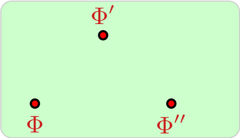

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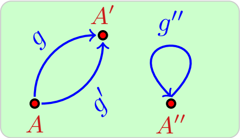

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Invariant information:

- $\pi_0\big(BG_{\mathrm{con}}(U)\big) = \Omega^1(U,\mathfrak{g})/C^\infty(U,G)$ ("gauge orbit space")

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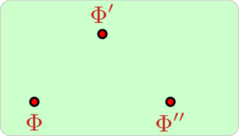

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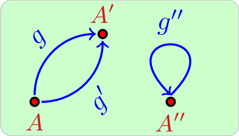

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- $\pi_1\big(BG_{\mathrm{con}}(U), A\big) = \{g \in C^\infty(U,G) : A = A \triangleleft g\}$ (stabilizers/“loops”)

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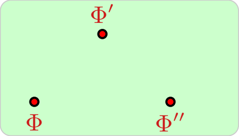

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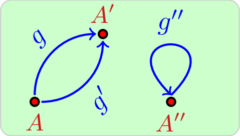

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$$BG_{\mathrm{con}}(U) \ = \begin{cases} \underline{\text{Obj:}} & A \in \Omega^1(U,\mathfrak{g}) \\ \underline{\text{Mor:}} & A \xrightarrow{g \in C^\infty(U,G)} A \triangleleft g = g^{-1}Ag + g^{-1}\mathrm{d}g \end{cases}$$

Invariant information:

- $\pi_0\big(BG_{\mathrm{con}}(U)\big) = \Omega^1(U,\mathfrak{g})/C^\infty(U,G)$ ("gauge orbit space")
- $\pi_1\big(BG_{\mathrm{con}}(U), A\big) = \{g \in C^\infty(U,G) : A = A \triangleleft g\}$ (stabilizers/"loops")

! $\mathbf{Grpd}$ is a 2-category (or model category) with weak equivalences the categorical equivalences $\Rightarrow$ need for higher (or derived) functors!

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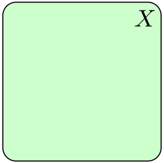

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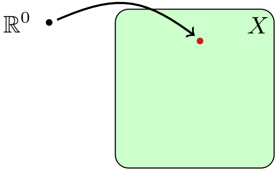

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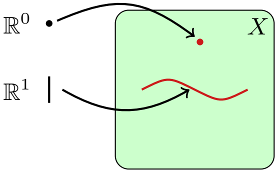

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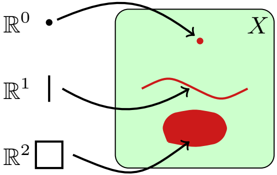

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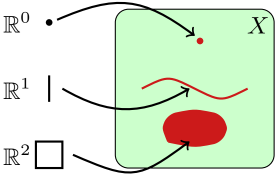

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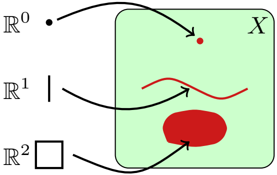

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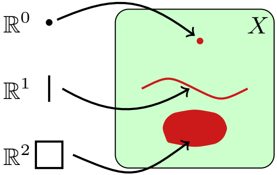

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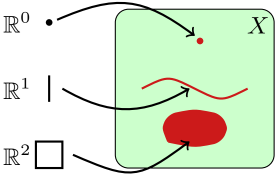

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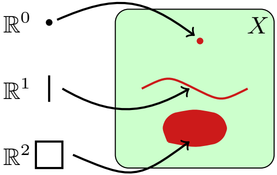

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! *Main observation:* Classical observables in a gauge theory are described by dg-algebras that are only homotopy-coherently commutative!

Higher structures in AQFT

Homotopy AQFTs: Definition and basic properties

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Smooth normalized cochain algebras on (a diagram $X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Stacks}$ of) stacks leads to homotopy AQFTs of this type.

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Def: The solution complex is defined as the (linear) derived critical locus of the action S, i.e. the following homotopy pullback in $\mathbf{Ch}_{\mathbb{K}}$

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Rem: Interpretation of $\mathfrak{Sol}(M)$ in terms of BRST/BV formalism from physics

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Towards descent in homotopy AQFT

- The inclusion $\overline{\mathbf{Loc}_\diamond} \hookrightarrow \overline{\mathbf{Loc}}$ of "diamonds" defines Quillen adjunction

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☹ Technically super (really, super!) complicated! Needs better technology for dealing with derived adjunctions, maybe ∞-categories à la Joyal/Lurie?

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$$\mathrm{ext} \,:\, \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}_\diamond}) \rightleftarrows \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}}) \,:\, \mathrm{res}$$
- $\diamond$ Homotopical descent condition: $\mathfrak{A} \in \mathbf{AQFT}_\infty(\overline{\mathbf{Loc}})$ is homotopy j-local iff derived counit $\widetilde{\epsilon}_{\mathfrak{A}} : \mathbb{L}\mathrm{ext}\,\mathrm{res}\,\mathfrak{A} \xrightarrow{\sim} \mathfrak{A}$ is weak equivalence
- $\diamond$ *Main question:* Is this condition fulfilled in examples, e.g. linear Yang-Mills?

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Then the derived extension $\mathbb{L}j_!\,\mathfrak{A}(M) = \mathrm{Sing}(M) \overset{\mathbb{L}}{\otimes} A$ at $M \in \mathbf{Man}_m$ is given by derived higher Hochschild chains on $\mathrm{Sing}(M)$ with values in A.

Toy-model: $\mathbb{R}$-Chern-Simons theory on 2-surfaces

◇ Linear observables for flat $\mathbb{R}$-connections on oriented 2-manifold M are

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$\Rightarrow$ This toy-model is homotopy j-local for $j : \overline{\mathbf{Disk}_2}^{\max} \to \overline{\mathbf{Man}_2}^{\max}$

Conclusions and outlook

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- What I would like to understand in the nearer future:
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  - ? Toy-models of non-linear homotopy AQFTs from derived algebraic geometry [Lurie; Calaque,Pantev,Toën,Vaquié,Vezzosi; Pridham]