On the time-slice axiom in 2d conformal AQFT

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Joint work w/ M. Benini and L. Giorgetti [arXiv:2111.01837]

Orthogonal categories and AQFTs

- **Def:** An orthogonal category $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ is a category $\mathbf{C}$ with a symmetric and $\circ$-stable subset

$$\perp \; = \Big\{ M_1 \xrightarrow{f_1} M' \xleftarrow{f_2} M_2 \Big\} \subseteq \mathrm{Mor}\,\mathbf{C} \; {}_t\!\times_t \; \mathrm{Mor}\,\mathbf{C} \, .$$

An AQFT on $\overline{\mathbf{C}}$ is a $\perp$-commutative functor $\mathfrak{A} : \mathbf{C} \longrightarrow \mathbf{Alg} \subseteq {}^{*}\mathbf{Alg}_{\mathsf{As}}\big(\underset{\uparrow \, \mathbf{ISMCat}}{\mathbf{T}}\big)$.

$$\begin{tikzcd}
\mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \arrow[r, "\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)"] \arrow[d, "\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)"'] \arrow[dr, phantom, "\forall\, f_1 \perp f_2"] & \mathfrak{A}(M')^{\otimes 2} \arrow[d, "\mu_{M'}"] \\
\mathfrak{A}(M')^{\otimes 2} \arrow[r, "\mu_{M'}^{\mathrm{op}}"'] & \mathfrak{A}(M')
\end{tikzcd}$$

- **Ex:** $\mathbf{Loc}_m$ w/ $\perp$ = causally disjoint

⇝ m-dim. LCQFTs satisfying Einstein causality

but NOT necessarily time-slice

Nice features of this formalism

Prop: The assignment $\overline{\mathbf{C}} \longmapsto \mathbf{AQFT}(\overline{\mathbf{C}})$ of AQFT categories can be upgraded to a 2-functor $\mathbf{AQFT} : \mathbf{OrthCat}^{op} \longrightarrow \mathbf{Cat}$.

Prop: Denote by $\mathbf{AQFT}(\overline{\mathbf{C}})^{W} \subseteq \mathbf{AQFT}(\overline{\mathbf{C}})$ the full subcategory of AQFTs satisfying time-slice for a set of maps $W \subseteq \mathrm{Mor}\,\mathbf{C}$, i.e.

$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\cong} \mathfrak{A}(M')$ is iso for all $(f : M \to M') \in W$.

The localization functor $L : \overline{\mathbf{C}} \longrightarrow \overline{\mathbf{C}}[W^{-1}] = \big(\mathbf{C}[W^{-1}] , L_\ast(\perp_{\mathbf{C}})\big)$ induces an equivalence of categories

$$L^\ast : \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}]) \xrightarrow{\sim} \mathbf{AQFT}(\overline{\mathbf{C}})^{W} .$$

In simple words: The time-slice axiom can be "solved" by computing the localized category $\mathbf{C}[W^{-1}]$.

Known cases where localizations have been computed

1.) Haag-Kastler type AQFTs on M [Benini, Dappiaggi, AS]

$$\mathrm{COpen}(M)[\mathrm{Cauchy}^{-1}] \simeq \mathrm{COpen}(M)^{D} = \{ U \subseteq M \; : \; DU = U \}$$

Cauchy development

2.) The "RCE category" [Bruinsma, Fewster, AS]

$$\left(\begin{tikzcd} & M_+ \arrow[dl] \arrow[dr] & \\ M & & M_h \\ & M_- \arrow[ul] \arrow[ur] & \end{tikzcd}\right)[\mathrm{Cauchy}^{-1}] \simeq B\mathbb{Z} = \begin{tikzcd} \bullet \arrow[loop above, "\mathbb{Z}"] \end{tikzcd}$$

the generator $1 \in \mathbb{Z}$ gives $rce_{(M,h)}$!

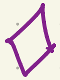
3.) 1d LCQFT [Bruinsma or AS, don't remember]

$$\mathrm{Loc}_1[\mathrm{Cauchy}^{-1}] \simeq B\mathbb{R} \Rightarrow \mathrm{AQFT}(\overline{\mathrm{Loc}_1})^{\mathrm{Cauchy}} \cong \{ A \in \mathrm{Alg} \text{ w/ } \mathbb{R}\text{-action } \boxed{\alpha : \mathbb{R} \to \mathrm{Aut}(A)} \}$$

time-evolution

Main Theorem of Today [Benini, Giorgetti, AS]

Let $\mathsf{Cloc}_2 :=$ { Obj: oriented and time-oriented Lorentzian 2-mfds M
that are globally hyperbolic and connected
Mor: orientation and time-orientation preserving embeddings
$f : M \longrightarrow M'$ s.t. $f(M) \subseteq M'$ is causally convex
and $f^*(g') = \Omega^2 g$ for some $\Omega^2 \in C^\infty(M, \mathbb{R}^{>0})$

and $i : \mathsf{C}_2^{D,\mathrm{skl}} \xrightarrow{\subseteq} \mathsf{Cloc}_2$ the full subcategory on the two objects given by the 2d Minkowski spacetime  and the flat cylinder .

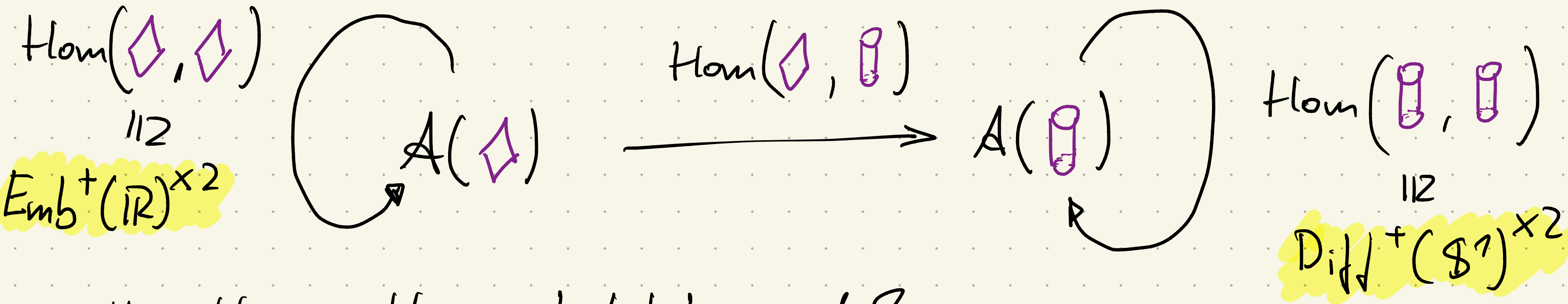
Then the inclusion functor i admits a left adjoint

$$L : \mathsf{Cloc}_2 \longrightarrow \mathsf{C}_2^{D,\mathrm{skl}}$$

that exhibits $\mathsf{C}_2^{D,\mathrm{skl}}$ as a localization of Cloc_2 at Cauchy.

Corollary

Each 2d conformal AQFT satisfying time-slice $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{CLoc}_2})^{\text{Cauchy}}$ is completely determined by only two algebras w/ a $\perp$-commutative action:



We call this the skeletal model.

Rem: The skeletal model is easy to compute:

Just restrict the functor $\mathfrak{A} : \mathbf{CLoc}_2 \longrightarrow \mathbf{Alg}$ to $\mathbf{C}_2^{\text{Disk}} \subseteq \mathbf{CLoc}_2$.

Reconstructing $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{CLoc}_2})^{\text{Cauchy}}$ from a skeletal model is much more involved, but can be done via operadic Kan extensions.

Proof sketch for main theorem:

(1) Global conformal embedding theorems [Finster, Müller ; Monclair]

Each $M \in \mathbf{Cloc}_2$ is isomorphic to causally convex open subset $U \subseteq$ 

$\Rightarrow$ $\mathbf{Cloc}_2$ is equivalent to full subcategory $\mathbf{C}_2 \subseteq \mathbf{Cloc}_2$ on such U's

(2) $\mathbf{Cloc}_2$-morphisms $f : U \subseteq$  $\longrightarrow U' \subseteq$  , $(x^+, x^-) \longmapsto \big(f^+(x^+), f^-(x^-)\big)$

extend uniquely to Cauchy developments $Df : DU \subseteq$  $\longrightarrow DU' \subseteq$ 

$\Rightarrow$ We get a functor $D : \mathbf{C}_2 \longrightarrow \mathbf{C}_2^D := \{ U \subseteq$  $: DU = U \}$

(3) Because $f : U \to U'$ is Cauchy iff $Df : DU \to DU'$ is iso, D is a localization.

(4) $\mathbf{C}_2^{\mathrm{Disk}} \subseteq \mathbf{C}_2^D$ is equivalence by a simple conformal geometry argument.

Summing up:

$$\begin{tikzcd} \mathbf{Cloc}_2 \arrow[rrr, bend left, "\text{Localization } L \text{ at Cauchy}"] & \mathbf{C}_2 \arrow[l, "\sim", "\subseteq"'] \arrow[r, shift left, "D"] & \mathbf{C}_2^D \arrow[l, shift left, "\subseteq"] & \mathbf{C}_2^{\mathrm{Disk}} \arrow[l, "\sim", "\subseteq"'] \end{tikzcd}$$

$\square$

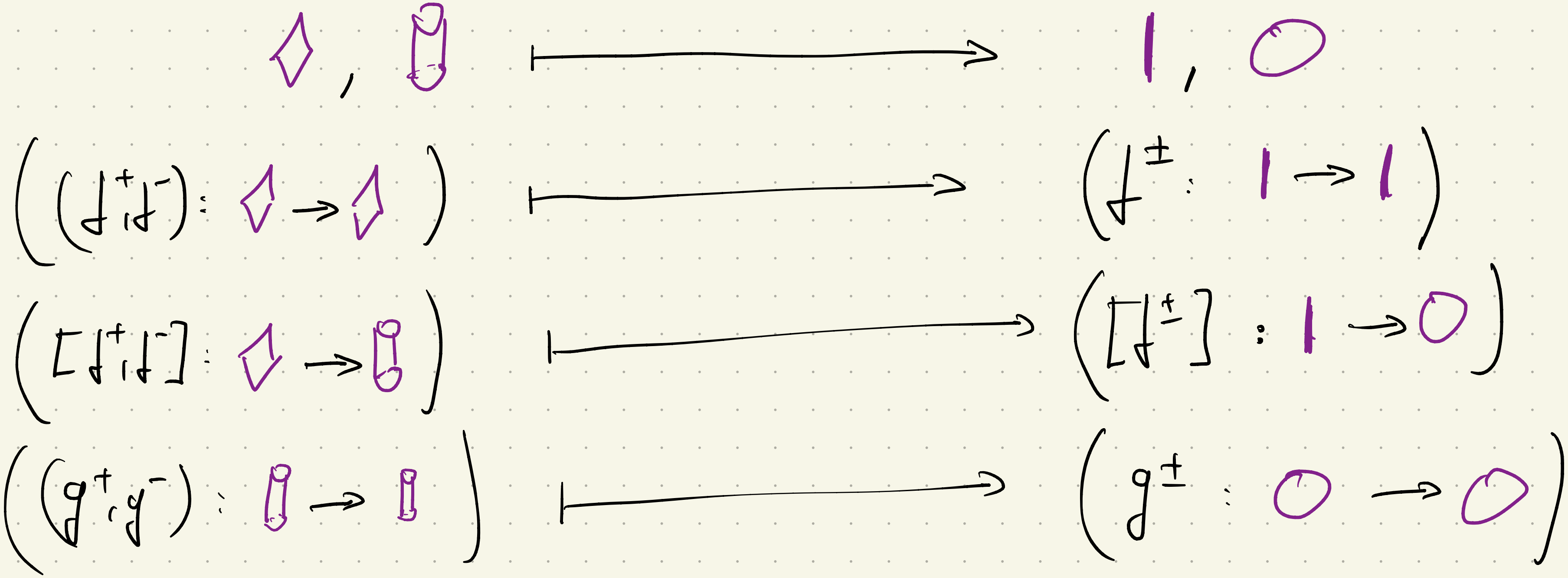
Application: Chiralization adjunction

- Wanted: Extract chiral components of $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{CLoc}_2})^{\text{Cauchy}}$.

  These should be chiral conformal AQFTs, i.e. theories on $\mathbf{Man}_1^{\text{or}}$ w/ $\perp$ = disjointness.

- Using skeletal models, there exists an obvious candidate:

$$\pi_\pm \;:\; \overline{\mathbf{C}_2^{D,\text{skl}}} \longrightarrow \overline{\mathbf{Man}_1^{\text{skl}}}$$



Thm: [Benini, Giorgetti, AS] Suppose that Alg is complete. The functor $\Pi_{\pm}^{*} : \mathbf{AQFT}(\overline{\mathbf{Man}_1^{\mathrm{shL}}}) \longrightarrow \mathbf{AQFT}(\overline{\mathbf{C}_2^{D,\mathrm{shL}}})$ is fully faithful and it admits a right adjoint (chiralization functor)

$$\Pi_{\pm *} : \mathbf{AQFT}(\overline{\mathbf{C}_2^{D,\mathrm{shL}}}) \longrightarrow \mathbf{AQFT}(\overline{\mathbf{Man}_1^{\mathrm{shL}}}) .$$

A model for $\Pi_{\pm *}$ is given by taking invariants of the opposite chirality

$$\Pi_{\pm *}(\mathfrak{A})(\,|\,) = \mathfrak{A}(\diamond)^{\mathrm{inv}_{\mp}} \leftarrow \text{invariants under } \mathrm{Emb}^+(\mathbb{R}) \text{ acting on } x^{\mp}$$

$$\Pi_{\pm *}(\mathfrak{A})(\bigcirc) = \mathfrak{A}(\,▯\,)^{\mathrm{inv}_{\mp}} \leftarrow \text{invariants under } \mathrm{Diff}^+(\mathbb{S}^1) \text{ acting on } x^{\mp}$$

Rem: That's a (locally covariant) generalization and formalization of Rehren's chiral observables.

Example: The Abelian current

- Solutions on $M \in \mathrm{CLoc}_2$: $\quad \mathrm{Sol}(M) = \{ j \in \Omega^1(M) \, : \, dj = d{*}j = 0 \}$
- Linear observables :

$$\mathcal{L}(M) = \frac{\Omega^1_c(M)}{d\, C^\infty_c(M) \oplus {*}d\, C^\infty_c(M)} \cong \underbrace{\frac{\Omega^+_c(M)}{d^+ C^\infty_c(M)}}_{\substack{\text{self-dual} \\ {*}\alpha = \alpha}} \oplus \underbrace{\frac{\Omega^-_c(M)}{d^- C^\infty_c(M)}}_{\substack{\text{anti-self-dual} \\ {*}\beta = -\beta}}$$

w/ suitable Poisson structure.

[similar to "F-model" of Fewster, Lang]

- Chiralization of $\mathcal{A} = \mathrm{CCR}(\mathcal{L})$:

$$\Pi_{\pm *}(\mathcal{A})(I) \cong \mathrm{CCR}\big(C^\infty_c(\mathbb{R}), \tau_{\text{usual}} \big)$$

← usual chiral current

$\Pi_{\pm *}(\mathcal{A})(\bigcirc) \cong \mathrm{CCR}\big(C^\infty(S^1), \tau_{\text{usual}} \big) \otimes \mathrm{Sym}\big(H^{\pm}_{c}($  $) \big)$ ← topological observables in opposite chirality

(anti-)self-dual cohomology

Topological invariants

- Taking further invariants, we get a functor

$$\mathrm{AQFT}\big(\overline{\mathsf{C}_2^{D,\mathrm{shL}}}\big) \xrightarrow{\pi_{\perp *}} \mathrm{AQFT}\big(\overline{\mathsf{Man}_1^{\mathrm{shL}}}\big) \xrightarrow{\text{Invariants}} \mathrm{AQFT}\big(\overline{\mathsf{G}_\bullet}\big)$$

where $\mathsf{G}_\circ = (\bullet \longrightarrow *)$ w/ $\perp = \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\}$.

- To $\mathfrak{A} \in \mathrm{AQFT}\big(\overline{\mathsf{C}_2^{D,\mathrm{shL}}}\big)$, this functor assigns the algebra map

$$\underbrace{\mathfrak{A}(\lozenge)^{\underline{\mathrm{inv}}}}_{\text{commutative}} \longrightarrow \underbrace{\mathfrak{A}(\text{▯})^{\underline{\mathrm{inv}}}}_{\text{not necessarily commutative}} \longleftarrow \text{complete invariants under } \mathrm{Emb}^+(\mathbb{R})^{\times 2} \text{ or } \mathrm{Diff}^+(\mathbb{S}^1)^{\times 2}$$

Ex: For the Abelian current:

$$\mathbb{C} \xrightarrow{\text{unit}} \mathrm{Sym}\big(H^1_c(\text{▯})\big)$$

Question: Do there exist interesting examples with $\mathfrak{A}(\text{▯})^{\underline{\mathrm{inv}}}$ a non-commutative algebra?

End