Gauge theories in locally covariant quantum field theory

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3. GK-Kolloquium @ Department of Mathematics, University of Regensburg, April 30, 2015.

Based on published and ongoing joint work with various subsets of
$\mathbf{Col} := \{$C. Becker, M. Benini, C. Dappiaggi, T. P. Hack, R. J. Szabo$\}$

Motivation

- QFT is one of the cornerstones of theoretical/mathematical physics with many applications to particle physics, solid state physics, cosmology, ...
- For a deeper understanding one has to address the question "What is a QFT?", i.e. we have to develop mathematical axioms for QFT.
- Prominent examples are:
  - Atiyah-Segal topological/conformal QFT (formalizes the Schrödinger picture)
    - ! Had a strong impact on pure mathematics, because TQFTs can be used to study invariants of manifolds.
  - Haag-Kastler local algebraic QFT (formalizes the Heisenberg picture)
    - ! Captures essential aspects of *physically relevant QFTs*, such as local d.o.f. and compatibility with the causal structure on Minkowski spacetime.
  - **Locally covariant QFT = algebraic QFT + Lorentz geometry**
    - ! Captures essential physical aspects of general relativity (geometry and causality on Lorentz manifolds). It is able to describe local and topological d.o.f.!
- Any axiomatic framework is as good as its examples! My goal is to study carefully the question: **How well does gauge theory fit into LCQFT?**

Outline

1. Quick introduction to locally covariant QFT

2. The geometry of gauge theories

3. Abelian quantum Yang-Mills theory: Construction, properties and problems

4. Homotopy theory in gauge theories

5. Towards a new framework: Homotopy locally covariant QFT

6. Concluding remarks

Quick introduction to locally covariant QFT

Basic physical idea

Locally covariant QFT is obtained by combining quantum theory with certain aspects of classical general relativity.

◇ A spacetime is a globally hyperbolic Lorentz manifold M.

As a first step, we shall neglect dynamical aspects of gravity (gravitons) and backreaction, so a QFT lives on a spacetime but it does not influence it.

(I) It is a priori not clear in which spacetime M we live, hence a QFT should be democratic and treat all of them on the same footing.

(II) In quantum theory, observables which can be measured in experiments are described by an abstract $*$-algebra A (sometimes assumed to be C^*).

◇ Combining (I) + (II): A QFT should be a mapping

$$\mathfrak{A} : \{ \text{ all spacetimes } \} \longrightarrow \{ \text{ all } *\text{-algebras } \} \ ,$$
$$M \quad \longmapsto \quad \mathfrak{A}(M) = \text{“QFT observables in } M\text{”}$$

NB: This assignment is too arbitrary! In particular, for a sub-spacetime $N \subseteq M$, the algebras $\mathfrak{A}(N)$ and $\mathfrak{A}(M)$ can be completely different.

$\Rightarrow$ **More structure is needed!**

The role of category theory

◇ Recall that a category C is a class of objects $\mathrm{Ob}(\mathsf{C})$ together with a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(C, C')$, for any pair of objects C, C'.

Morphisms can be composed in an associative way via a composition map $\circ : \mathrm{Hom}_{\mathsf{C}}(C', C'') \times \mathrm{Hom}_{\mathsf{C}}(C, C') \to \mathrm{Hom}_{\mathsf{C}}(C, C'')$ and there are identity morphisms $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{C}}(C, C)$.

Ex:
- The category of spacetimes Loc has as objects all globally hyperbolic Lorentz manifolds (oriented, time-oriented and of fixed dimension m) and as morphisms all causal isometric open embeddings $f : M \to M'$.
- The category of algebras Alg has as objects all $*$-algebras and as morphisms all $*$-algebra homomorphisms $\kappa : A \to A'$.

! Notice that sub-spacetime relations $N \subseteq M$ are encoded as morphisms $\iota_{M,N} : N \to M$ in Loc.

◇ We can improve our definition of a QFT by demanding it to be a functor

$$\mathfrak{A} : \mathsf{Loc} \longrightarrow \mathsf{Alg} \quad .$$

From this we get for any spacetime M an algebra $\mathfrak{A}(M)$ and for any spacetime embedding $f : M \to M'$ an algebra map $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$.

Brunetti-Fredenhagen-Verch axioms

- Not any functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$ will describe a physically reasonable QFT, so one has to impose additional axioms!
- The original BFV-axioms are:

(L) *Locality axiom:* For any Loc-morphism $f : M \to M'$, the Alg-morphism $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$ is monic (i.e. injective).

(C) *Causality axiom:* For any Loc-diagram $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ such that the images of f_1 and f_2 are causally disjoint, the commutator

$$[\,-\,,\,-\,]_{\mathfrak{A}(M)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(M)$$

is zero.

(T) *Time-slice axiom:* For any Loc-morphism $f : M \to M'$ such that the image contains a Cauchy surface of M', the Alg-morphism $\mathfrak{A}(f)$ is isomorphism.

Ex:
- ✓ Quantized Klein-Gordon theory $(-\Box + m^2 + \xi R)\phi = 0$ [BFV,...].
- ✓ Formal interacting (scalar) QFTs [Brunetti,Fredenhagen,Hollands,Rejzner,...].
- ✓ After slight modifications, also free quantized Dirac theory [Sanders,...].
- ↯ Gauge theories, even the Abelian ones. This is bad! Why that?

The geometry of gauge theories

Bundles, gauge fields, and all that

- Gauge theory was born by globalizing and generalizing Maxwell's theory of electromagnetism.
- As a first step, we choose a structure group G, which in Maxwell's theory is $G = \mathbb{T} = U(1)$ and in particle physics some product with $SU(n)$'s.
- A gauge field configuration on a manifold M is a pair $\mathcal{A} = (P, \omega)$, where $P \stackrel{\pi}{\longrightarrow} M$ is a principal G-bundle over M and ω a connection on P.

NB:
- The bundle P describes the topological sector, e.g. magnetic monopole charge for $G = \mathbb{T}$ or the instanton sector for $G = SU(n)$.
- The connections on P describe fluctuations around the topological sector, e.g. photons for $G = \mathbb{T}$ or gluons for $G = SU(3)$.

- A gauge transformation is an arrow $g : \mathcal{A} \to \mathcal{A}'$ given by a vertical principal G-bundle isomorphism $g : P \to P'$ such that $g^*(\omega') = \omega$ under pull-back.

NB:
- If $M \simeq \mathbb{R}^m$, then all bundles are trivial $P \simeq M \times G$ and a gauge field configuration is simply an element $\mathcal{A} \in \Omega^1(M, \mathfrak{g})$ (called gauge potential).
- In this case gauge transformations reduce to the usual well-known formula $\mathcal{A}' = g^{-1}\, \mathcal{A}\, g + g^{-1}\, \mathrm{d}g$, where $g \in C^\infty(M, G)$.

Gauge orbit spaces (a.k.a. coarse moduli spaces)

◇ I will now follow the (too naive) folklore that 'only gauge classes matter'.

◇ Mathematically, this is done by forming the gauge orbit space

$$\mathrm{Conf}_G(M) := \big\{\text{all gauge fields } \mathcal{A} \text{ on } M\big\}\big/\sim \quad ,$$

where the quotient is by all gauge transformations.

◇ For a general structure group G, the geometric structure of $\mathrm{Conf}_G(M)$ is complicated, so let me fix in the following $G = \mathbb{T}$, i.e. Abelian gauge theory.

◇ $\widehat{H}^2(M) := \mathrm{Conf}_{\mathbb{T}}(M)$ is Abelian group and we have group homomorphisms:

- *Curvature/field strength:* $\mathrm{curv} : \widehat{H}^2(M) \to \Omega^2(M)$
- *Characteristic class/magnetic charge:* $\mathrm{char} : \widehat{H}^2(M) \to H^2(M,\mathbb{Z})$

NB: Chern-Weil theory: De Rham class of $\mathrm{curv}(\mathcal{A}) \in \Omega^2(M)$ is equal to $\mathrm{char}(\mathcal{A})$ (modulo torsion). Hence, we have a commutative diagram

$$\begin{array}{ccc} \widehat{H}^2(M) & \xrightarrow{\ \mathrm{curv}\ } & \Omega^2_{\mathbb{Z}}(M) \\ {\scriptstyle \mathrm{char}}\big\downarrow & & \big\downarrow{\scriptstyle [\,-\,]_{\mathsf{dR}}} \\ H^2(M,\mathbb{Z}) & \longrightarrow & H^2_{\mathrm{free}}(M,\mathbb{Z}) \end{array}$$

A fancy diagram of exact sequences

- Studying carefully further aspects of the gauge orbit space one finds that our small diagram can be extended to the diagram of exact sequences ($* = 2$):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^{*-1}(M,\mathbb{R})}{H^{*-1}_{\mathrm{free}}(M,\mathbb{Z})} & \longrightarrow & \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathbb{Z}}(M)} & \xrightarrow{\mathrm{d}} & \mathrm{d}\Omega^{*-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & H^{*-1}(M,\mathbb{T}) & \xrightarrow{\kappa} & \widehat{H}^*(M) & \xrightarrow{\mathrm{curv}} & \Omega^*_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow & & \\
0 & \longrightarrow & H^*_{\mathrm{tor}}(M,\mathbb{Z}) & \longrightarrow & H^*(M,\mathbb{Z}) & \longrightarrow & H^*_{\mathrm{free}}(M,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

- Physical interpretation:
  - Gauge classes of connections on trivial bundle
  - Gauge classes of flat connections
  - Characteristic classes/magnetic charges
  - Curvatures/field strengths

Quick look at abstract differential cohomology

Def: A differential cohomology theory is a functor $\widehat{H}^* : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Ab}^{\mathbb{Z}}$ to $\mathbb{Z}$-graded Abelian groups together with four natural transformations $(\mathrm{curv}, \mathrm{char}, \iota, \kappa)$ that fits into the natural diagram of exact sequences on the previous slide.

Thm: [Simons,Sullivan; Bär,Becker] Differential cohomology theories exist (e.g. Cheeger-Simons) and are unique up to a unique natural isomorphism.

Prop: (The geometry of differential cohomology) [Becker,AS,Szabo]

(i) A differential cohomology theory can be promoted to a functor $\widehat{H}^* : \mathsf{Man}^{\mathrm{op}} \to \mathsf{FrAb}^{\mathbb{Z}}$ to $\mathbb{Z}$-graded Abelian Frechét-Lie groups, such that the natural diagram of exact sequences becomes a diagram in $\mathsf{FrAb}^{\mathbb{Z}}$.

(ii) Isomorphism types: $\widehat{H}^k(M) \simeq \mathbb{T}^{b_{k-1}} \times H^k(M,\mathbb{Z}) \times \mathrm{d}\Omega^{k-1}(M)$, where b_{k-1} is the $k-1$-th Betti number of M.

Rem:

1. Physically, the factor $\mathbb{T}^{b_{k-1}}$ describes the Aharonov-Bohm phases, $H^k(M,\mathbb{Z})$ the magnetic charges and $\mathrm{d}\Omega^{k-1}(M)$ the linear field strength perturbations.
2. Differential cohomology is interesting for different degrees:

$k = 1$ σ-model with target space $\mathbb{T}$

$k = 2$ Abelian $\mathbb{T}$-gauge theory

$k \geq 3$ Higher Abelian gauge theories on $k-2$-gerbes (important for string theory)

Abelian quantum Yang-Mills theory: Construction, properties and problems

Construction of Abelian quantum Yang-Mills theory

◇ The quantization of differential cohomology is rather technical, so I can only give a sketch. Details are available in my paper with Becker and Szabo.

1. Take a differential cohomology theory $\widehat{H}^* : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Ab}^{\mathbb{Z}}$. Fix some degree $k \geq 1$ and induce $\widehat{H}^k : \mathsf{Loc}^{\mathrm{op}} \to \mathsf{Ab}$ to the spacetime category Loc.
2. On Loc we have a natural equation of motion, namely Maxwell's equation $\mathrm{MW} := \delta \circ \mathrm{curv} : \widehat{H}^k \Rightarrow \Omega^{k-1}$ with codifferential $\delta : \Omega^k \Rightarrow \Omega^{k-1}$.
3. Characterize the solution groups $\mathrm{Sol}^k := \mathrm{Ker}(\mathrm{MW})$, which turns out to be a subfunctor of $\widehat{H}^k$ that takes values in Abelian Frechét-Lie groups and fits into a nice diagram of exact sequences.
4. Use Peierls' method to obtain from Maxwell's Lagrangian a natural Poisson-Frechét manifold structure on the solution groups Sol^k.
5. Take as classical observables the Poisson $*$-algebras generated by smooth group characters on Sol^k (smooth Pontryagin duality).
6. Quantize these Poisson $*$-algebras to C^*-algebras by using techniques from CCR-quantization of presymplectic Abelian groups.

Thm: [Becker,AS,Szabo]

For any $k \geq 1$, the above construction yields a functor $\mathfrak{A}^k : \mathsf{Loc} \to C^*\mathsf{Alg}$, which satisfies the causality and time-slice axiom of the BFV-axioms.

What about the locality axiom?

Thm: [Becker,AS,Szabo; earlier results by Benini,Dappiaggi,Hack,AS]

(a) The functor $\mathfrak{A}^k : \mathsf{Loc} \to C^*\mathsf{Alg}$ has a subfunctor of the form
$\mathfrak{A}^k_{\mathrm{top}} := \mathfrak{CCR} \circ \big(H^k(-,\mathbb{Z})^* \oplus H^{m-k}(-,\mathbb{R})^*\big) : \mathsf{Loc} \longrightarrow C^*\mathsf{Alg}$.

(b) For any Loc-morphism $f : M \to M'$ the following are equivalent:

1. $\mathfrak{A}^k(f) : \mathfrak{A}^k(M) \to \mathfrak{A}^k(M')$ is monic.
2. $f_* : H^k(M,\mathbb{Z})^* \oplus H^{m-k}(M,\mathbb{R})^* \to H^k(M',\mathbb{Z})^* \oplus H^{m-k}(M',\mathbb{R})^*$ is monic.

(c) Unless $(m,k) = (2,1)$, the functor $\mathfrak{A}^k$ violates the locality axiom.

◇ Physical interpretation:

(a) $\mathfrak{A}^k_{\mathrm{top}}$ is a topological QFT, measuring the topological content of Abelian Yang-Mills theory given by electric charges and magnetic charges.

(b) + (c) It is precisely due to topological charges that the locality axiom is violated!
This violation can be understood as a topological obstruction for extending 'charged' gauge field configurations from M to M'. E.g.

(i) For nontrivial electric charge $Q_{\mathrm{el}} \neq 0$, the static gauge potential $A \sim Q_{\mathrm{el}}/r$ on $\mathbb{R}^3 \setminus \{0\}$ does not extend to a solution of Maxwell's equation on $\mathbb{R}^{3+1}$.

(ii) For nontrivial magnetic charge (i.e. Chern class) $Q_{\mathrm{mag}} \neq 0$, the $\mathbb{T}$-bundle $P \to \mathbb{R}^3 \setminus \{0\}$ does not extend to $\mathbb{R}^3$.

[For experts: The differential cohomology presheaf is not flabby (or at least c-soft).]

Local-to-global properties

◇ **Conceptual problem:** Due to violations of the locality axiom we can not effectively compare and relate observables via $\mathfrak{A}^k(f) : \mathfrak{A}^k(M) \to \mathfrak{A}^k(M')$ whenever M is topologically non-trivial!

? Can we compare local and global physics in a different way?

! Introduce gluing axiom! *Heuristically:*

For any M, the global observable algebra $\mathfrak{A}^k(M)$ should be "determined by" the local algebras $\mathfrak{A}^k(U_\alpha)$ in a suitable open cover $\{U_\alpha \to M\}$.

◇ There are (at least) two possible precise definitions:

– *Additivity axiom:* $\mathfrak{A}^k(M) \simeq \bigvee_\alpha \mathfrak{A}^k(U_\alpha)$ [studied in LCQFT by Fewster,Verch]

– *Cosheaf axiom:* $\mathfrak{A}^k(M) \xleftarrow{\simeq} \operatorname{colim}\Big(\coprod_{\alpha,\beta} \mathfrak{A}^k(U_{\alpha\beta}) \rightrightarrows \coprod_{\alpha} \mathfrak{A}^k(U_\alpha)\Big)$ [stronger!]

Prop: For $k \geq 2$, the functor $\mathfrak{A}^k$: Loc $\to C^*$Alg satisfies neither the cosheaf nor the additivity axiom.

NB:

– *Physical interpretation:* Gauge invariant observables can not be glued!

– This can be seen already for configurations: Gauge classes can not be glued!

– In mathematical terminology: Differential cohomology is **not** a sheaf and as a consequence its quantization is **not** a cosheaf.

Homotopy theory in gauge theories

Why is gauge theory different to, say, scalar field theory?

◇ Scalar field configurations form a sheaf $\mathfrak{F} := C^\infty(\,-\,,\mathbb{R}) : \mathrm{Man}^{\mathrm{op}} \to \mathrm{Sets}$, i.e. for any open cover $\{U_\alpha\}$ of a manifold M we have the gluing law

$$\mathfrak{F}(M) \xrightarrow{\simeq} \lim\Big(\prod_\alpha \mathfrak{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathfrak{F}(U_{\alpha\beta})\Big)$$

◇ Gauge orbit spaces do not form a sheaf, so there is no gluing law!

◇ If we **DO NOT** take orbit spaces, then gauge field configurations form a stack, which is the same thing as a homotopy sheaf of groupoids [Hollander].

◇ A homotopy sheaf $\mathfrak{G} : \mathrm{Man}^{\mathrm{op}} \to \mathrm{Gpds}$ satisfies the "homotopy gluing law"

$$\mathfrak{G}(M) \xrightarrow{\sim} \mathrm{holim}\Big(\prod_\alpha \mathfrak{G}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathfrak{G}(U_{\alpha\beta}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{\alpha,\beta,\gamma} \mathfrak{G}(U_{\alpha\beta\gamma}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \;\cdots\Big)$$

NB: The "homotopy gluing law" is **not** as crazy as it looks like!

Already Dirac has used some variant of it to construct the Abelian magnetic monopole by "gluing up to a gauge transformation"

$$A_\beta|_{U_{\alpha\beta}} - A_\alpha|_{U_{\alpha\beta}} = g_{\alpha\beta}\,\mathrm{d}g_{\alpha\beta}^{-1} \quad , \quad g_{\beta\gamma}|_{U_{\alpha\beta\gamma}}\, g_{\alpha\gamma}^{-1}|_{U_{\alpha\beta\gamma}}\, g_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = 1$$

homotopy sheaf = "gluing up to a gauge transformation"

Configurations and observables in contractible manifolds

- The homotopy sheaf property suggest the following strategy:
  1. Formulate gauge field configurations and observables in contractible manifolds.
  2. Extend via homotopy (co)limits to all manifolds.
- On a contractible manifold M, the groupoid of gauge field configurations may be described by the simplicial set

$$\Omega^1(M,\mathfrak{g}) \Longleftarrow C^\infty(M,G)\times\Omega^1(M,\mathfrak{g}) \Lleftarrow C^\infty(M,G)^{\times 2}\times\Omega^1(M,\mathfrak{g}) \Lleftarrow \cdots$$

- Gauge field observables are then suitable functions on this simplicial set, i.e. a cosimplicial algebra

$$\mathcal{O}\big(\Omega^1(M,\mathfrak{g})\big) \Longrightarrow \mathcal{O}\big(C^\infty(M,G)\times\Omega^1(M,\mathfrak{g})\big) \Rrightarrow \mathcal{O}\big(C^\infty(M,G)^{\times 2}\times\Omega^1(M,\mathfrak{g})\big) \Rrightarrow \cdots$$

Rem:
- For making a suitable choice of functions $\mathcal{O}$ one has to equip the configurations with a simplicial (locally convex) manifold structure.
- Applying dual Dold-Kan, the cosimplicial algebra gives rise to a dg-algebra, which can be 'linearized' via the van-Est map to the Chevalley-Eilenberg dg-algebra corresponding to infinitesimal gauge transformations. This is the starting point of the BRST/BV-formalism [Fredenhagen,Rejzner in LCQFT].

! Notice that our approach has the advantage that it describes finite gauge transformations! (These are important for gluing!)

Extension to generic manifolds

- On the previous slide we have seen that:
  - gauge field configurations may be described by a functor $\mathfrak{C} : \mathsf{Man}_{©}^{\mathrm{op}} \to \mathsf{sSet}$.
  - gauge field observables may be described by a functor $\mathfrak{O} : \mathsf{Man}_{©} \to \mathsf{cAlg}$.
- **Wanted:** Extension of these functors to all manifolds by computing homotopy (co)limits over suitable covers.
- **Problem:** It is really hard to compute explicitly these homotopy (co)limits!
- We [Benini,AS,Szabo] have tackled this problem and made explicit calculations for Abelian gauge theory, which is much easier because there is a description in terms of chain complexes of Abelian groups.

  The details are technical, so I can not explain them here and refer to our recent preprint.
- **Main results:** Homotopy limits produce the correct global gauge field configurations (i.e. differential cohomology in chain complex homology) and homotopy colimits produce the correct global gauge field observables.

! This solves the problems which shown up in the ordinary universal algebra (i.e. colimit) construction of [Dappiaggi,Lang; Fewster,Lang]!

(missing flat connections, violation of Dirac charge quantization condition, . . .)

Towards a new framework:
Homotopy locally covariant QFT

A working definition of holCQFT

Def: A homotopy locally covariant QFT is a functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{X}$ to some suitable model category of 'algebras' (maybe dgAlg or scAlg) which satisfies:

- **(G)** *Homotopy cosheaf axiom:* $\mathfrak{A}$ is a homotopy cosheaf. (We can glue observables!)
- **(wL)** *Weak locality axiom:* For any Loc-morphism $f : M \to M'$ such that M is contractible, the morphism $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$ is monic up to homotopy.
- **(C)** *Causality axiom:* For any Loc-diagram $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$ such that the images of f_1 and f_2 are causally disjoint, the commutator

$$[\,-\,,\,-\,]_{\mathfrak{A}(M)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(M)$$

  is zero up to homotopy.
- **(T)** *Time-slice axiom:* For any Cauchy Loc-morphism $f : M \to M'$, the morphism $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$ is an isomorphism up to homotopy.

NB: These structures have not yet been explored in detail, but they seem to be essential for formulating gauge theories. Open problems/Future work:

- Give precise definition of holCQFT. (What do we mean by up to homotopy?)
- Is (Abelian) quantum Yang-Mills theory a holCQFT?
- Can we do interesting model-independent studies in holCQFT? (E.g. Relative Cauchy evolution, automorphism groups, spin-statistics theorem, ...)

Concluding remarks

Concluding remarks

- I hope that I could convince you that **Abelian** quantum Yang-Mills theory is not yet as well understood as people always claim.
- In particular, there is a deep conflict between the mathematical structure of gauge theories and the axiomatic framework of locally covariant QFT:

$$\textbf{gauge theory} + \textbf{LCQFT} = \lightning$$

- The source of this problem is that ordinary LCQFT does not capture important structural aspects of gauge theories ("stacky" geometry of configurations spaces; homotopical algebra of observables; ...).
- Because of the immense relevance of gauge theories in physics and mathematics, it is unavoidable for us to generalize our techniques of LCQFT in order to make them compatible with gauge theories.
- Our proposed framework exactly goes in this direction:

$$\textbf{homotopy theory} + \textbf{LCQFT} = \textbf{holCQFT} \overset{?}{\ni} \text{gauge theories}$$