The inhomogeneous Klein-Gordon field: A new standard model for LCQFT!?!

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Motivation

- The Klein-Gordon field is by far the best studied model of QFT on curved spacetimes, so let us call it the standard model of LCQFT.
- This is not a coincidence, but a consequence of the simplicity of this model:
  - linear configuration space $C^\infty(M)$
  - linear dynamics $P(\phi) = (\Box + m^2)\phi = 0$
  - linear solution space $\mathrm{Sol}(M) = \{\phi \in C^\infty(M) : P(\phi) = 0\}$
- This linear structure is also shared by other models, e.g. the free Dirac field, but it is **not** present in gauge and/or interacting theories.
- Since interacting theories are too complicated (at the moment), we shall look at the simplest generalization of linear theories:
  - affine configuration space
  - affine dynamics
  - affine solution space

Goal: Obtain a full understanding of the simplest affine model, which is the inhomogeneous Klein-Gordon field $P(\phi) = (\Box + m^2)\phi + J = 0$.

$\Rightarrow$ A new and more involved standard model for LCQFT!?!?!

Outline

1. Locally covariant QFT in one slide
2. The inhomogeneous Klein-Gordon field à la BDS
3. Automorphism group of the $\mathfrak{PhSp}_p$-functor
4. Composition property of the $\mathfrak{PhSp}_p$-functor?
5. A better functor: A case for using Poisson algebras
6. The quantized theory
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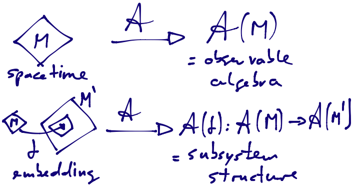
Locally covariant QFT in one slide

Locally covariant QFT: The shortest crashcourse ever!

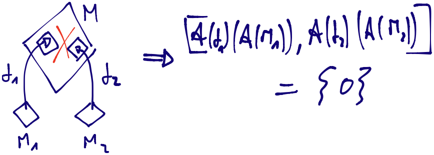
◇ What should a QFT do?

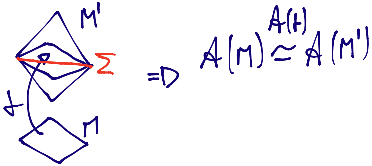


Def: A locally covariant QFT (LCQFT) (à la Brunetti, Fredenhagen, Verch) is a covariant functor $\mathfrak{A} : \mathsf{Loc} \to (C)^*\mathsf{Alg}$, such that

(i) if $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ are causally disjoint, then $\mathfrak{A}(f_1)[\mathfrak{A}(M_1)]$ and $\mathfrak{A}(f_2)[\mathfrak{A}(M_2)]$ commute as subalgebras of $\mathfrak{A}(M)$ *(causality axiom)*

(ii) if $f : M \to M'$ is Cauchy morphism (i.e. $f[M] \subseteq M'$ contains Cauchy surface), then $\mathfrak{A}(f)$ is isomorphism *(time-slice axiom)*





The inhomogeneous Klein-Gordon field à la BDS

Inhomogeneous field theory → Affine field theory

- In [Benini,Dappiaggi,Schenkel: AHP 2013] we have proposed to consider inhomogeneous field theories as a special class of affine field theories.
- To see how this works, take your favorite linear field theory, i.e.
  - a geometric category Geo,
  - a contravariant functor of vector bundle sections $\mathfrak{C}^\infty : \mathsf{Geo} \to \mathsf{Vec}$, and
  - a natural transformation by Green-hyperbolic operators $P^{\mathrm{lin}} : \mathfrak{C}^\infty \Rightarrow \mathfrak{C}^\infty$.
- The inhomogeneous theory is then given by the following data:
  - the enriched geometric category GeoSrc with objects being tuples $(M, J \in \mathfrak{C}^\infty(M))$ and compatible morphisms,
  - the contravariant functor $\mathfrak{A}^\infty : \mathsf{GeoSrc} \to \mathsf{Aff}$ obtained by applying the forgetful functor $\mathsf{Vec} \to \mathsf{Aff}$ to $\mathfrak{C}^\infty$, and
  - the natural transformation by affine Green-hyperbolic operators $P : \mathfrak{A}^\infty \Rightarrow \mathfrak{C}^\infty$ given by $P_{(M,J)}(\,\cdot\,) = P^{\mathrm{lin}}_M(\,\cdot\,) + J$.
- To any such inhomogeneous theory one can assign a covariant functor $\mathfrak{PhSp} : \mathsf{GeoSrc} \to \mathsf{PreSymp}$ and a locally covariant QFT $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp} : \mathsf{GeoSrc} \to {}^*\mathsf{Alg}$. [BDS]

$\rightsquigarrow$ I will now show the details for the inhomogeneous Klein-Gordon field.

The inhomogeneous Klein-Gordon field: Kinematics

- For the Klein-Gordon field a suitable geometric category is

  Loc: $\mathrm{Obj}(\mathsf{Loc})$ are oriented, time-oriented and glob. hyp. Lorentzian manifolds.

  $\mathrm{Mor}(\mathsf{Loc})$ are orientation and time-orientation preserving isometric embeddings, such that the image is causally compatible and open.

- A multiplet of $p \in \mathbb{N}$ homogeneous Klein-Gordon fields is described by:
  - contravariant functor $\mathfrak{C}_p^\infty : \mathsf{Loc} \to \mathsf{Vec}$, with $\mathfrak{C}_p^\infty(M) = C^\infty(M, \mathbb{R}^p)$ and $\mathfrak{C}_p^\infty(f : M_1 \to M_2) = f^* : C^\infty(M_2, \mathbb{R}^p) \to C^\infty(M_1, \mathbb{R}^p)$,
  - natural transformation $\mathsf{KG} : \mathfrak{C}_p^\infty \Rightarrow \mathfrak{C}_p^\infty$

$$\mathsf{KG}_M : \mathfrak{C}_p^\infty(M) \to \mathfrak{C}_p^\infty(M) \ , \quad \phi \mapsto \mathsf{KG}_M(\phi) = (\Box_M + m^2)\phi$$

- Following the general recipe, we get:

$\mathsf{LocSrc_p}$: $\mathrm{Obj}(\mathsf{LocSrc_p})$ are tuples (M, J), where M in Loc and $J \in \mathfrak{C}_p^\infty(M)$.

$\mathrm{Mor}(\mathsf{LocSrc_p})$ are all morphisms $f : M_1 \to M_2$ in Loc, such that $\mathfrak{C}_p^\infty(f)(J_2) = f^*(J_2) = J_1$.

  - contravariant functor $\mathfrak{A}_p^\infty : \mathsf{LocSrc_p} \to \mathsf{Vec}$, with $\mathfrak{A}_p^\infty(M, J) = C^\infty(M, \mathbb{R}^p)$ and $\mathfrak{A}_p^\infty(f : (M_1, J_1) \to (M_2, J_2)) = f^* : C^\infty(M_2, \mathbb{R}^p) \to C^\infty(M_1, \mathbb{R}^p)$,
  - natural transformation $\mathsf{P} : \mathfrak{A}_p^\infty \Rightarrow \mathfrak{C}_p^\infty$

$$\mathsf{P}_{(M,J)} : \mathfrak{A}_p^\infty(M, J) \to \mathfrak{A}_p^\infty(M, J) \ , \quad \phi \mapsto \mathsf{P}_{(M,J)}(\phi) = (\Box_M + m^2)\phi + J$$

The inhomogeneous Klein-Gordon field: Phase space

- Consider the following covariant functor (affine dual of $\mathfrak{A}_p^\infty$):
  - $\mathfrak{A}_p^{\infty,\dagger} : \mathsf{LocSrc}_p \to \mathsf{Vec}$, with $\mathfrak{A}_p^{\infty,\dagger}(M,J) = C_0^\infty(M,\mathbb{R}^{p+1})$ and $\mathfrak{A}_p^{\infty,\dagger}(f:(M_1,J_1)\to(M_2,J_2)) = f_\ast : C_0^\infty(M_1,\mathbb{R}^{p+1}) \to C_0^\infty(M_2,\mathbb{R}^{p+1})$

  and its subfunctor $\mathfrak{Triv}_p : \mathsf{LocSrc}_p \to \mathsf{Vec}$, with

$$\mathfrak{Triv}_p(M,J) = \Big\{ a\otimes e_{p+1} \in C_0^\infty(M,\mathbb{R}^{p+1}) : \int \mathrm{vol}_M\, a = 0 \Big\}$$

- The quotient $\mathfrak{A}_p^{\infty,\dagger}/\mathfrak{Triv}_p : \mathsf{LocSrc}_p \to \mathsf{Vec}$ has a further subfunctor $\mathsf{P}^\dagger(\mathfrak{C}_{p,0}^\infty)$ describing the equation of motion, where

$$\mathsf{P}^\dagger(\mathfrak{C}_{p,0}^\infty)(M,J) = \mathsf{P}^\dagger_{(M,J)}\big(C_0^\infty(M,\mathbb{R}^p)\big) \subseteq \mathfrak{A}_p^{\infty,\dagger}(M,J)/\mathfrak{Triv}_p(M,J)$$

- The quotient $\big(\mathfrak{A}_p^{\infty,\dagger}/\mathfrak{Triv}_p\big)/\mathsf{P}^\dagger(\mathfrak{C}_{p,0}^\infty) : \mathsf{LocSrc}_p \to \mathsf{Vec}$ can be enriched to a covariant functor $\mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ by defining

$$\sigma_{(M,J)}\big([\varphi],[\psi]\big) = \int \mathrm{vol}_M\, \langle \varphi_V, \mathsf{E}_M(\psi_V)\rangle$$

Thm:
(i) $\mathfrak{PhSp}_p$ satisfies the causality property and the time-slice axiom.

(ii) $\mathfrak{PhSp}_p$ has a nontrivial subfunctor $\mathfrak{N}_p : \mathsf{LocSrc}_p \to \mathsf{Vec}$ describing the kernel of the presymplectic structures.

(iii) $\mathfrak{N}_p$ is naturally isomorphic to $\mathbb{R} : \mathsf{LocSrc}_p \to \mathsf{Vec}$, with $\mathbb{R}(M) = \mathbb{R}$.

Automorphism group of the $\mathfrak{PhSp}_p$-functor

Generalities

Def: An endomorphism of a covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{D}$ is a natural transformation $\eta : \mathfrak{F} \Rightarrow \mathfrak{F}$, i.e. a collection of morphisms $\{\eta_C : \mathfrak{F}(C) \to \mathfrak{F}(C)\}$, such that for any morphism $f : C_1 \to C_2$ in C

$$\begin{array}{ccc} \mathfrak{F}(C_1) & \xrightarrow{\mathfrak{F}(f)} & \mathfrak{F}(C_2) \\ {\scriptstyle \eta_{C_1}} \downarrow & & \downarrow {\scriptstyle \eta_{C_2}} \\ \mathfrak{F}(C_1) & \xrightarrow{\mathfrak{F}(f)} & \mathfrak{F}(C_2) \end{array}$$

The collection of all endomorphisms of $\mathfrak{F}$ is denoted by $\mathrm{End}(\mathfrak{F})$.

Def: An automorphism of a covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{D}$ is a natural transformation $\eta : \mathfrak{F} \Rightarrow \mathfrak{F}$, such that all η_C are isomorphisms. The collection of all automorphisms is the group $\mathrm{Aut}(\mathfrak{F})$.

NB: For a (quantum) field theory functor, e.g. $\mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$, the group $\mathrm{Aut}(\mathfrak{PhSp}_p)$ describes global symmetries of the theory on the functorial level. This is comparable to the global gauge group of Minkowski AQFT. See [Fewster: RMP 2013] for details on automorphism groups.

Finding automorphisms of $\mathfrak{PhSp}_p$

- Naively: Look at the action functional

$$S_{(M,J)}[\phi] = \int \mathrm{vol}_M \left(-\frac{1}{2}\langle \partial_\mu \phi, \partial^\mu \phi \rangle + \frac{m^2}{2}\langle \phi, \phi \rangle + \langle J, \phi \rangle \right)$$

$\rightarrow$ $O(p)$ symmetry is broken, for $m = 0$ it remains $\phi \mapsto \phi + \mu$.

Expectation: $\mathrm{Aut}(\mathfrak{PhSp}_p) = \{\mathrm{id}_{\mathfrak{PhSp}_p}\}$ for $m \neq 0$ and $\mathbb{R}^p$ for $m = 0$.

- Rather mysteriously (explanation later) we obtain the following:

Prop: For any covariant functor $\mathfrak{F} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ there exists a faithful homomorphism $\eta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathfrak{F})$ given by $\eta(\sigma) = \{\sigma\, \mathrm{id}_{\mathfrak{F}(M,J)}\}$, $\sigma \in \mathbb{Z}_2 = \{-1, +1\}$.

Prop: For $m = 0$ there exists a faithful homomorphism $\eta : \mathbb{Z}_2 \times \mathbb{R}^p \to \mathrm{Aut}(\mathfrak{PhSp}_p)$ given by, for all $[(\varphi, \alpha)] \in \mathfrak{PhSp}_p(M,J)$,

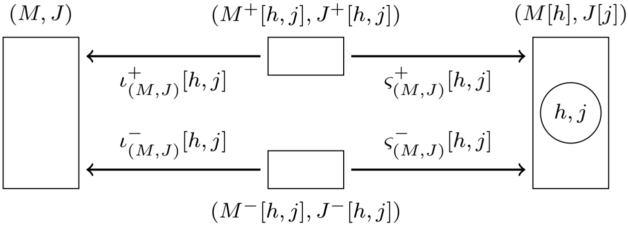
$$\eta(\sigma,\mu)_{(M,J)}\big([(\varphi,\alpha)]\big) = \Big[\Big(\sigma\,\varphi, \sigma\,\alpha + \sigma \int \mathrm{vol}_M\, \langle \varphi, \mu \rangle\Big)\Big]$$

$\Rightarrow$ $\mathrm{Aut}(\mathfrak{PhSp}_p)$ contains $\mathbb{Z}_2$ for $m \neq 0$ and $\mathbb{Z}_2 \times \mathbb{R}^p$ for $m = 0$!

?? Are these all automorphisms?

The relative Cauchy evolution: A tool for computing Aut

◇ For any globally hyperbolic perturbation (h, j) of (M, J) we have a diagram



◇ Since $\mathfrak{PhSp}_p$ satisfies the time-slice axiom, we can define

$$\mathrm{rce}_{(M,J)}[h,j] = \mathfrak{PhSp}_p(\iota^-_{(M,J)}[h,j]) \circ \mathfrak{PhSp}_p(\varsigma^-_{(M,J)}[h,j])^{-1} \circ \mathfrak{PhSp}_p(\varsigma^+_{(M,J)}[h,j]) \circ \mathfrak{PhSp}_p(\iota^+_{(M,J)}[h,j])^{-1}$$

Prop:

$$\mathcal{T}_{(M,J)}[h]\big([(\varphi,\alpha)]\big) := \frac{d}{ds}\mathrm{rce}_{(M,J)}[s\,h,0]\big([(\varphi,\alpha)]\big)|_{s=0} = -\Big[\Big(\mathrm{KG}'_{M[h]}(\mathsf{E}_M(\varphi)), \int \mathrm{vol}_M\, \frac{g^{ab}\,h_{ab}}{2}\,\langle J, \mathsf{E}_M(\varphi)\rangle\Big)\Big]$$

$$\mathcal{J}_{(M,J)}[j]\big([(\varphi,\alpha)]\big) := \frac{d}{ds}\mathrm{rce}_{(M,J)}[0,s\,j]\big([(\varphi,\alpha)]\big)|_{s=0} = -\Big[\Big(0, \int \mathrm{vol}_M\, \langle j, \mathsf{E}_M(\varphi)\rangle\Big)\Big]$$

How can we compute $\mathrm{End}(\mathfrak{PhSp}_p)$ and $\mathrm{Aut}(\mathfrak{PhSp}_p)$?

- This is a quite complicated task! We have to characterize how $\eta \in \mathrm{End}(\mathfrak{PhSp}_p)$ interplays with (potential) symmetries of (M, J), general morphisms in LocSrc_p and the rce.

Lem: Let η be any endomorphism and $f : (M, J) \to (M, J)$ an endomorphism of (M, J). Then $\eta_{(M,J)} \circ \mathfrak{PhSp}_p(f) = \mathfrak{PhSp}_p(f) \circ \eta_{(M,J)}$.

"Functor endomorphisms commute with symmetries!"

Lem: $\eta_{(M,J)} \circ \mathrm{rce}_{(M,J)}[h, j] = \mathrm{rce}_{(M,J)}[h, j] \circ \eta_{(M,J)}$.

"Functor endomorphisms commute with rce, and in particular its derivatives!"

Lem: Let $\eta, \eta' \in \mathrm{End}(\mathfrak{PhSp}_p)$ be such that $\eta_{(M,J)} = \eta'_{(M,J)}$ for some (M, J).

(i) If $f : (L, J_L) \to (M, J)$ is morphism, then $\eta_{(L,J_L)} = \eta'_{(L,J_L)}$.

(ii) If $f : (M, J) \to (N, J_N)$ is Cauchy morphism, then $\eta_{(N,J_N)} = \eta'_{(N,J_N)}$.

(iii) $\eta_{(L,J_L)} = \eta'_{(L,J_L)}$ for any (L, J_L), such that the Cauchy surfaces of L are oriented diffeomorphic to those of $M|_O$, with $O \in \mathcal{O}(M)$.

Thm: Every $\eta \in \mathrm{End}(\mathfrak{PhSp}_p)$ is uniquely determined by its component on any object (M, J).

$\Rightarrow$ Strategy: Look for endomorphisms $\mathrm{End}(\mathfrak{PhSp}_p(M_0, 0))$ ($(M_0, 0)$ Minkowski spacetime) that commute with rce and Poincaré transformations!

Main result for $\mathrm{End}(\mathfrak{PhSp}_p)$ and $\mathrm{Aut}(\mathfrak{PhSp}_p)$

Theorem

For the functor $\mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ *we have*

$$\mathrm{End}(\mathfrak{PhSp}_p) = \mathrm{Aut}(\mathfrak{PhSp}_p) \simeq \begin{cases} \mathbb{Z}_2 & , \quad \text{for } m \neq 0 \; , \\ \mathbb{Z}_2 \times \mathbb{R}^p & , \quad \text{for } m = 0 \; . \end{cases}$$

There immediately pop up questions:

- Why is the automorphism group too big?
- Is this some sort of "hidden symmetry"?
- Or is it a flaw in our description of the inhomogeneous Klein-Gordon field?

What I want to show now is that it is indeed a flaw in our description!

In the construction of $\mathfrak{PhSp}_p$ we have *forgotten* that $[(\varphi,\alpha)]$ is supposed to describe functionals on the affine space of solutions to $\Box_M \phi + m^2\phi + J = 0$.

Re-introducing this piece of information will provide us with a better functor!

Composition property of the $\mathfrak{PhSp}_p$-functor?

The functor $\mathfrak{PhSp}_p$ is not so good: Reason 2

Let me take a multiplet of p inhomogeneous KG fields, split it into 2 pieces, treat them separately and afterwards compose the result. Do I get the same as when treating the original multiplet? Let us formalize this physical idea:

- "Splitting into 2 pieces" is done by the covariant functor $\mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{LocSrc}_q \times \mathsf{LocSrc}_{p-q}$ defined by $\mathfrak{Split}_{p,q}(M,J) = \big((M,J^q),(M,J^{p-q})\big)$ and $\mathfrak{Split}_{p,q}(f) = (f,f)$.
- "Treating them separately" is $\mathfrak{PhSp}_q \times \mathfrak{PhSp}_{p-q} : \mathsf{LocSrc}_q \times \mathsf{LocSrc}_{p-q} \to \mathsf{PreSymp} \times \mathsf{PreSymp}$.
- "Compose the result" is $\oplus : \mathsf{PreSymp} \times \mathsf{PreSymp} \to \mathsf{PreSymp}$ defined by $\big(V \oplus W, \sigma_{V\oplus W}\big)$ with $\sigma_{V\oplus W}\big((v,w),(v',w')\big) = \sigma_V(v,v') + \sigma_W(w,w')$.

$\Rightarrow$ We get another locally covariant field theory functor

$$\mathfrak{PhSp}_{p,q} := \oplus \circ \big(\mathfrak{PhSp}_q \times \mathfrak{PhSp}_{p-q}\big) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$$

Prop: The functors $\mathfrak{PhSp}_p$ and $\mathfrak{PhSp}_{p,q}$ are **not** naturally isomorphic. (Even more, they are not even unnaturally isomorphic.)

Reason: The null space of $\mathfrak{PhSp}_p$ is 1 dimensional and the one of $\mathfrak{PhSp}_{p,q}$ is 2D.

$\Rightarrow$ $\mathfrak{PhSp}_p$ is not a good description of inhomogeneous KG fields.

A better functor: A case for using Poisson algebras

The canonical Poisson algebras

- There is an obvious covariant functor $\mathfrak{CanPois} : \mathsf{PreSymp} \to \mathsf{PoisAlg}$:
  - $\mathfrak{CanPois}(V, \sigma_V)$ is the symmetric algebra $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ with the Poisson bracket defined by $\{v_1, v_2\}_{\sigma_V} = \sigma_V(v_1, v_2)$.
  - $\mathfrak{CanPois}(L : (V, \sigma_V) \to (W, \sigma_W))$ is defined by $\mathfrak{CanPois}(L)(v_1 \cdots v_k) = L(v_1) \cdots L(v_k)$.

Prop:

a) $\mathfrak{PA}_p := \mathfrak{CanPois} \circ \mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ satisfies the causality property and the time-slice axiom.

b) $\mathrm{Aut}(\mathfrak{PA}_p)$ contains a $\mathbb{Z}_2$ subgroup for $m \neq 0$ and a $\mathbb{Z}_2 \times \mathbb{R}^p$ subgroup for $m = 0$.

c) $\mathfrak{PA}_p$ violates the composition property, i.e. it is not isomorphic to

$$\mathfrak{PA}_{p,q} := \otimes \circ \left(\mathfrak{PA}_q \times \mathfrak{PA}_{p-q}\right) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$$

- As expected, taking canonical Poisson algebras does not solve the problems.

  However, the category of Poisson algebras is much richer than PreSymp and it allows us to construct improved Poisson algebras!

The improved Poisson algebras

Goal: Make the theory given by $\mathfrak{PA}_p := \mathfrak{CanPois} \circ \mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ remember that it came from functionals acting on affine solution spaces.

Def: The contravariant functor $\mathfrak{Sol}_p : \mathsf{LocSrc}_p \to \mathsf{Aff}$ is the subfunctor of $\mathfrak{A}_p^\infty$ defined by $\mathfrak{Sol}_p(M,J) := \{\phi \in C^\infty(M,\mathbb{R}^p) : P_{(M,J)}(\phi) = 0\}$.

- ◇ There is a natural pairing between the covariant functor $\mathfrak{PA}_p$ and the contravariant functor $\mathfrak{Sol}_p$ defined by, for all $[(\varphi,\alpha)] \in \mathfrak{PA}_p(M,J)$ and $\phi \in \mathfrak{Sol}_p(M,J)$,

$$\big\langle\big\langle [(\varphi,\alpha)],\phi\big\rangle\big\rangle_{(M,J)} := \Big(\int \mathrm{vol}_M\, \langle\varphi,\phi\rangle\Big) + \alpha$$

Lem: $\mathfrak{I}_p(M,J) := \big\{a \in \mathfrak{PA}_p(M,J) : \big\langle\big\langle a, \mathfrak{Sol}_p(M,J)\big\rangle\big\rangle_{(M,J)} = \{0\}\big\}$ is a Poisson ideal. It is equal to $\big\langle\big\{[(0,\alpha)] - \alpha : \alpha \in \mathbb{R}\big\}\big\rangle$.

Thm:

a) The quotient $\mathfrak{A}_p := \mathfrak{PA}_p / \mathfrak{I}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ satisfies the causality property and the time-slice axiom.

b) $\mathrm{End}(\mathfrak{A}_p) = \mathrm{Aut}(\mathfrak{A}_p) \simeq \begin{cases} \{\mathrm{id}_{\mathfrak{A}_p}\} & , \text{ for } m \neq 0 \,, \\ \mathbb{R}^p & , \text{ for } m = 0 \,. \end{cases}$

c) The composition property holds, i.e. $\mathfrak{A}_p$ is naturally isomorphic to $\mathfrak{A}_{p,q}$.

The quantized theory

The canonical quantum algebras

◇ Everybody knows the covariant functor $\mathfrak{CCR} : \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ associating the quantized field polynomial algebras to presymplectic vector spaces.

Prop:

a) $\mathfrak{PQ}_p := \mathfrak{CCR} \circ \mathfrak{PhSp}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the causality property and the time-slice axiom.

b) $\mathrm{Aut}(\mathfrak{PQ}_p)$ contains a $\mathbb{Z}_2$ subgroup for $m \neq 0$ and a $\mathbb{Z}_2 \times \mathbb{R}^p$ subgroup for $m = 0$.

c) $\mathfrak{PQ}_p$ violates the composition property, i.e. it is not isomorphic to

$$\mathfrak{PQ}_{p,q} := \otimes \circ \big(\mathfrak{PQ}_q \times \mathfrak{PQ}_{p-q}\big) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$$

⇒ The canonical quantum algebras $\mathfrak{PQ}_p$ do not give a satisfactory description of the multiplet of p inhomogeneous KG fields.

Require a suitable modification of $\mathfrak{PQ}_p$, which *remembers* the fact that it came from functionals on an affine solution space.

The improved quantum algebras

Def: a) A state space $\mathfrak{S}_p$ for $\mathfrak{PQ}_p$ is a contravariant functor $\mathfrak{S}_p : \mathsf{LocSrc}_p \to \mathsf{State}$, such that $\mathfrak{S}_p(M,J)$ is a state space for $\mathfrak{PQ}_p(M,J)$ and such that $\mathfrak{S}_p(f)$ is the restriction of the dual of $\mathfrak{PQ}_p(f)$.

b) An admissible state space $\mathfrak{S}_p$ for $\mathfrak{PQ}_p$ is a state space, such that for all $\omega \in \mathfrak{S}_p(M,J)$, $\omega\big([(0,\alpha)]\big) = \alpha$ and $\omega\big([(0,\alpha)]\,[(0,\beta)]\big) = \alpha\,\beta$.

Lem: (i) There exists a non-empty admissible state space. This is proven by using the pull-back techniques of [BDS].

(ii) $\mathfrak{J}^{\mathfrak{S}_p}(M,J) := \bigcap_{\omega\in\mathfrak{S}_p(M,J)} \ker \pi_\omega$ is a both-sided $*$-ideal. For any non-empty admissible state space it is equal to $\big\langle\big\{[(0,\alpha)] - \alpha : \alpha \in \mathbb{R}\big\}\big\rangle$.

Thm: The quotient $\mathfrak{Q}_p := \mathfrak{PQ}_p/\mathfrak{J}^{\mathfrak{S}_p} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the causality property and the time-slice axiom, i.e. it is a locally covariant QFT.

Conj: $\mathfrak{Q}_p$ has the correct automorphism group and satisfies the composition property. Hence, $\mathfrak{Q}_p$ is a more suitable description of a multiplet of p inhomogeneous KG fields than the functor $\mathfrak{PQ}_p$ proposed in [BDS].

Rem: It can be shown that $\mathfrak{Q}_p(M,J)$ is (noncanonically) isomorphic to the algebra of the homogeneous KG field $\mathfrak{PQ}_p^{\mathrm{lin}}(M)$. $\Rightarrow$ All the information about the sources is captured in the functorial structure, not in the individual algebras.

Summary and outlook

Summary and outlook

- I hope that I could convince you that the inhomogeneous Klein-Gordon field could serve as a new standard model for LCQFT.
- We have understood well how to construct the relevant algebras, which are **not** given by usual canonical quantization of (pre)symplectic vector spaces, but by a more complicated procedure.
- We also have understood many structural properties of the inhomogeneous Klein-Gordon field, like the existence of good classes of states, the automorphism group and the relative Cauchy evolution.
- Our results (modulo some modifications due to gauge invariance) also apply to $U(1)$-gauge theory and provide an instruction for how to improve the algebras derived in [BDS,BDHS].

Grazie per la vostra attenzione!